Jacobi Factors of Quasi-Homogeneous Plane Curve Singularities

Mikhail Mazin
(joint work with Eugene Gorsky)

Stony Brook University

21 October 2012

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Theorem (Beauville)

For a rational unibranched curve C its compactified Jacobian is homeomorphic to the direct product of compact spaces, the Jacobi factors $\overline{JC}_p$, $p \in Sing(C)$, which depend uncover on the analytic type of the singularities (C, p).

Jacobi Factors

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Let $\delta = \dim(\mathbb{C}[[t]]/R)$. Since $x \in C$ is a plane curve singularity, it follows that $t^{2\delta}\mathbb{C}[[t]] \subset R$.

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Definition

The *Jacobi factor* $\overline{JC}_x$ is the space of R-submodules $M \subset \mathbb{C}[[t]]$, such that $M \supset t^{2\delta}\mathbb{C}[[t]]$ and $\dim(\mathbb{C}[[t]]/M) = \delta$.
In other words, $\overline{JC}_x$ is isomorphic to the subvariety of the Grassmannian $Gr(V, \delta)$, consisting of subspaces invariant under R-action.

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Remark

Compactified Jacobian of a cuspidal elliptic curve C is the curve C itself. It is homeomorphic, but not isomorphic to $\mathbb{P}^1$.

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$$\dim C_\Delta = \sum_{j=0}^{m-1} |([a_j, a_j + n) \setminus \Delta)|.$$

where $(0 = a_0 < a_1 < \ldots < a_{m-1})$ is the m–basis of Δ.

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Therefore,

$$\dim C_\Delta = 2 + 2 + 0 = 4.$$

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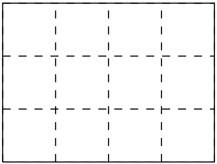

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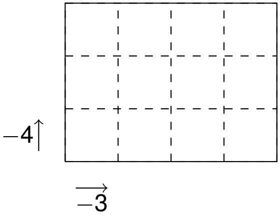

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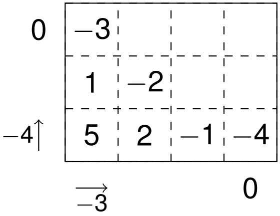

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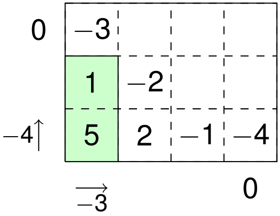

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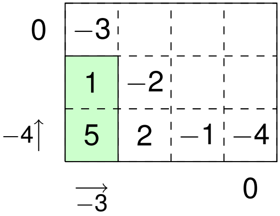
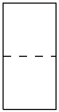


$D(\Delta) =$ 

Hilbert Scheme of Points in $\mathbb{C}^2$

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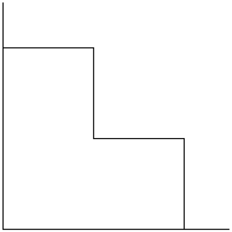

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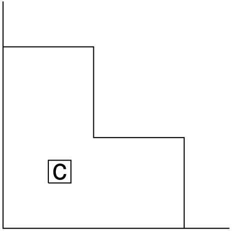

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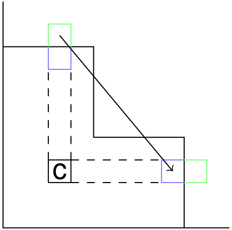

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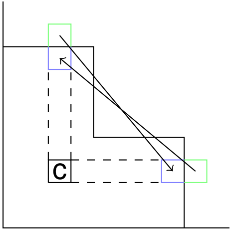

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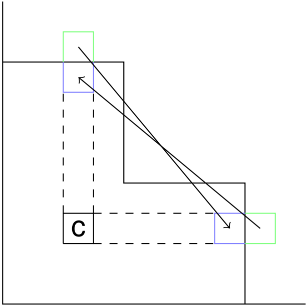

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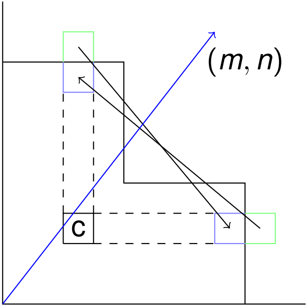

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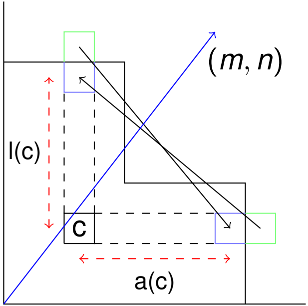

$$h^+_{\frac{m}{n}}(D) = \#\left\{c \in D \middle| \frac{a(c)}{l(c)+1} \leq \frac{m}{n} < \frac{a(c)+1}{l(c)}\right\}$$

Dimensions of Cells

Theorem

The dimension of the cell corresponding to the Young diagram D in the cell decomposition of $\mathrm{Hilb}^d(\mathbb{C}^2)$ *given by the subgroup* $T^{m,n}$ *is equal to*

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Theorem (E. Gorsky, M.M)

The dimension of the cell corresponding to the Young diagram D in the cell decomposition of the Jacobi factor of the singularity $\{x^m = y^n\}$ *is equal to*

$$\delta - h_+^{\frac{m}{n}}(D).$$

where $\delta = \frac{(m-1)(n-1)}{2}$ *is the* δ*-invariant of the singularity.*

Symmetry of Polynomials $c_{m,n}(q,t)$ 1

Motivated by the work of J. Haglund, we introduce the following polynomials:

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It motivates the following conjecture:

Conjecture (Symmetry of Polynomials $c_{m,n}(q,t)$)

The function $c_{m,n}(q,t)$ satifies the functional equation

$$c_{m,n}(q,t) = c_{m,n}(t,q).$$

Symmetry of Polynomials $c_{m,n}(q,t)$ 2

This symmetry is known if $m = n+1$. In this case it follows from some nontrivial relations on q,t-Catalan numbers. For $m = kn+1$ a similar statement was conjectured by N. Loehr. No bijective proof in any of these cases is known yet.

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The symmetry conjecture holds for $n \leq 3$.

In the proof we construct an explicit bijection exchanging the area and h_+ statistics.

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We also formulate the following weaker version of the symmetry conjecture:

Conjecture (Weak Symmetry)

The function $c_{m,n}(q,t)$ satifies the functional equation

$$c_{m,n}(q,1) = c_{m,n}(1,q).$$

Symmetry of Polynomials $c_{m,n}(q,t)$ 3

Theorem (J. Haglund; N. Loehr; E.Gorsky, M. M.)

The weak symmetry conjecture holds for $m = kn \pm 1$.

All these cases are proved by an explicit bijective construction.

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Proof.

$$P(t) = \sum t^{\delta - h_+(D)} = t^{\delta} c_{m,n}(1, t^{-1}) = t^{\delta} c_{m,n}(t^{-1}, 1) = \sum t^{|D|}.$$

□

Maps G_m and G_n

I will discuss the weak symmetry in few more details.

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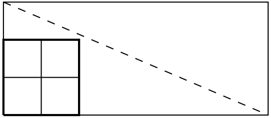
This result allows us to consider the map G_m from the set of diagrams below the diagonal to itself, sending $D(M)$ to a diagram with columns $g_m(a_i)$.

Example for Maps G_3 and G_7

$D(M)$



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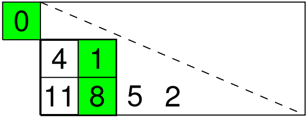
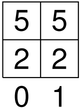


$G_3(M)$



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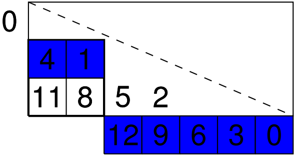
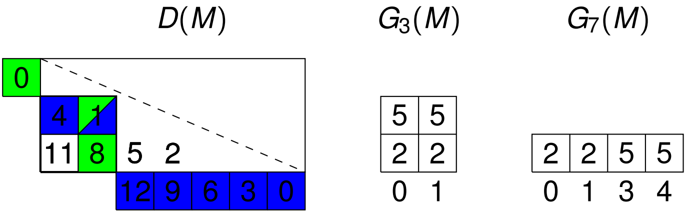


$G_7(M)$

2	2	5	5
0	1	3	4

Example for Maps G_3 and G_7



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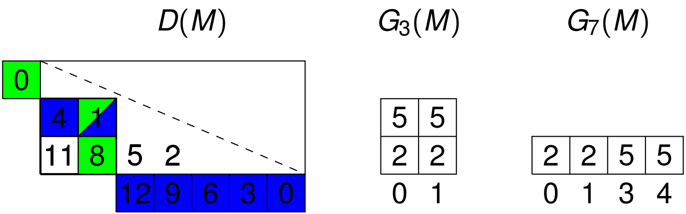

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The map G_m is a bijection for $m = kn \pm 1$.

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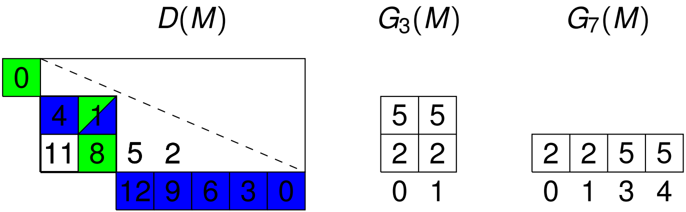

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The weak symmetry for $m = kn \pm 1$ follows from this theorem. Indeed,

$$\dim C_\Delta = |G_m(D(\Delta))|.$$

Thank you!