Mutations of Laurent Polynomials and Flat Families with Toric Fibers

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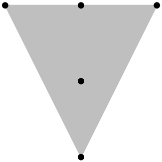
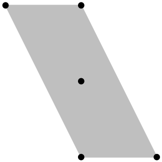
$\Delta(f_1) =$ 

$\Delta(f_2) =$ 

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The constant terms series of a Laurent polynomial $f \in \mathbb{C}[\mathbb{Z}^n]$ is the power series

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For $f = f_1 = x^{-1}y + 2y + xy + y^{-1}$ or
$f = f_2 = x^{-1}y + y + y^{-1} + y^{-1}x$,

$$C_f(t) = 1 + 4t^2 + 36t^4 + 400t^6 + 4900t^8 + \dots$$

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Let g be a nonzero Laurent polynomial in $z_1, \ldots, z_{n-1}$. The birational transformation

$$\phi_g \in \mathrm{Aut}(\mathbb{C}(z_1, \ldots, z_n)) \qquad \phi_g(z_i) = \begin{cases} z_i & \text{if } 1 \leq i < n \\ z_n/g & \text{if } i = n \end{cases}$$

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Example

Take $n = 2$, $x = z_1$, $y = z_2$, and $g = x + 1$.

$$\begin{aligned} \phi_g(f_1) &= \phi_g(x^{-1}y(x+1)^2 + y^{-1}) \\ &= \quad x^{-1}y(x+1) + y^{-1}(x+1) = f_2. \end{aligned}$$

Mutations continued

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Remark

Let ϕ_g be a simple mutation as above, and f a Laurent polynomial such that $\phi_g(f)$ is also a Laurent polynomial. Then $C_f(t) = C_{\phi(f)}(t)$.

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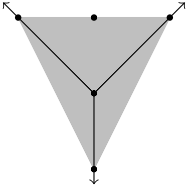

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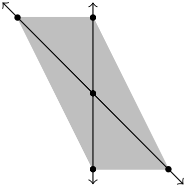

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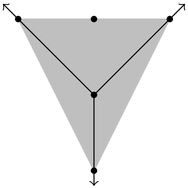

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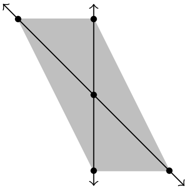

Note that $\mathbb{P}(1,1,2)$ *deforms to* $\mathbb{P}^1 \times \mathbb{P}^1$*!*

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- The family π has a natural fiberwise $(\mathbb{C}^*)^{n-1}$ action (where n is the dimension of the fibers of π).
- The family π is constructed using more general techniques developed by R. Vollmert and me.

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Let Δ *be a reflexive polytope with* Δ^* *normal, and* V *a smooth Fano variety. Then the following are equivalent:*

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- The conjecture is true in dimension two.
- If true, the above might be used to help classify higher dimensional Fano varieties.

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- Fano threefolds thus provide a good testing ground for the conjecture.
- Together with J. Christophersen, I have classified embedded degeneration of smooth Fano threefolds to toric Gorenstein Fano varieties for degrees ≤ 12.

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