A SHORT PROOF OF THE PRIME NUMBER THEOREM FOR ARITHMETIC PROGRESSIONS

IVAN SOPROUNOV

Abstract. We give a short proof of the Prime Number Theorem for arithmetic progressions following the ideas of recent Newman's short proof of the usual Prime Number Theorem.

1. Introduction

A few years ago D. J. Newman found a simple analytic proof of the Prime Number Theorem. It contains a very elegant proof of the fact that $\zeta(s)$ has no zeros in $\Re s = 1$, and a refined short derivation of the Prime Number Theorem from it. The proof presented by D. Zagier in [Z] only takes three pages since the author avoids unnecessary details. This makes the proof even more appealing.

It seems natural to have an analogous short proof of the Prime Number Theorem for arithmetic progressions. It turns out that the same arguments can be applied to this problem. The proof we present here is similar to the one in [Z]. As for the length, it takes approximately the same number of pages, even though we put more detail in it. However, the proof relies on the fact that $L(1,\chi) \neq 0$ for any non-principal character χ, whose proof we omit. This can be found in any proof of the Dirichlet theorem (see for example [BSh]).

2. The Prime Number Theorem for arithmetic progressions

Theorem 2.1. *Let $\pi(x,q)$ denote the number of all primes p no greater than x, congruent to a mod q, for $a, q \in \mathbb{N}$ such that $\gcd(a,q) = 1$. Then*

$$\pi(x,q) \sim \frac{1}{\phi(q)} \frac{x}{\log x},$$

where $\phi(q)$ is the Euler function.

Following the original paper [Z] we will give a proof in a series of steps. Let χ be a Dirichlet character modulo q and χ_0 the principal character. We consider the following functions:

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \theta_q(x) = \phi(q) \sum_{\substack{p \leq x \\ p \equiv a\ (q)}} \log p,$$

$$\phi(s,\chi) = \sum_p \frac{\chi(p)\log p}{p^s}, \quad \Phi_q(s) = \sum_\chi \phi(s,\chi), \quad \Phi_{q,a}(s) = \sum_\chi \overline{\chi(a)}\phi(s,\chi).$$

Claim 1. For $\Re s > 1$ we have $L(s,\chi) = \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1}$. In particular $L(s,\chi_0) = \prod_{p \nmid q} \left(1 - p^{-s}\right)^{-1} = \prod_{p|q} \left(1 - p^{-s}\right) \zeta(s)$.

The proof of the claim repeats the proof of the corresponding statement in [Z], taking into account the multiplicativness of χ.

Claim 2. For any non-principal χ the functions $L(s,\chi)$ and $L(s,\chi_0) - \frac{\phi(q)}{q}\frac{1}{s-1}$ extend holomorphically to $\Re s > 0$.

Proof. We are using the standard partial summation argument:

$$\sum_{n\le x}\frac{\chi(n)}{n^s} = \frac{A(x)}{x^s} + s\int_1^x \frac{A(t)\,dt}{t^{s+1}},$$

where $A(x) = \sum_{n\le x}\chi(n)$. Note that if $\chi \neq \chi_0$ then $A(x)$ is bounded thus

$$L(s,\chi) = s\int_1^\infty \frac{A(t)\,dt}{t^{s+1}}$$

represents a holomorphic function for $\Re s > 0$. For the principal character we use the representation in Claim 1 which gives a meromorphic extension to $\Re s > 0$ with a simple pole at $s = 1$ and the residue $\prod_{p|q}(1-p^{-1}) = \phi(q)/q$. □

Claim 3. $\theta_q(x) = O(x)$.

This follows immediately from statement III of [Z] since

$$\theta_q(x) \le \phi(q)\sum_{p\le x}\log p = \phi(q)\theta(x) = O(x).$$

Claim 4. For any χ, $L(s,\chi)$ has no zeros in $\Re s \ge 1$.

Proof. Consider the function $\mathcal{L}(s) = \prod_\chi L(s,\chi)$. We will show that $\mathcal{L}(s)$ has no zeros on $\Re s = 1$. We already know from the proof of the Dirichlet Theorem [BSh] that $L(1,\chi) \neq 0$ for any non-principal χ. By Claim 2 $L(s,\chi_0)$ has a simple pole at $s = 1$ and hence so does $\mathcal{L}(s)$. Suppose $\mathcal{L}(s)$ has a zero of order $\mu \ge 0$ at $s = 1 + i\alpha$ for $\alpha \neq 0$. Denote the order of zero at $s = 1 + 2i\alpha$ by ν. Observe that $\mathcal{L}(s)$ is a real-valued function for s real. Indeed,

$$\overline{\mathcal{L}(s)} = \prod_\chi \overline{L(s,\chi)} = \prod_\chi L(s,\overline{\chi}) = \mathcal{L}(s).$$

Therefore $\mathcal{L}(s)$ has zeros of orders μ and ν at $s = 1 - i\alpha$ and $s = 1 - 2i\alpha$, respectively.

Now for any character χ we have for $\Re s > 1$

$$\begin{aligned} -\frac{L'(s,\chi)}{L(s,\chi)} &= -\sum_p \frac{d}{ds}\left(\log(1-\chi(p)p^{-s})^{-1}\right) = \sum_p \frac{\chi(p)p^{-s}\log p}{1-\chi(p)p^{-s}} \\ &= \sum_p \frac{\chi(p)\log p}{p^s - \chi(p)} = \sum_p \frac{\chi(p)\log p}{p^s} + \sum_p \frac{\chi^2(p)\log p}{p^s(p^s-\chi(p))} \qquad (2.1) \\ &= \phi(s,\chi) + h(s,\chi), \end{aligned}$$

where the function $h(s,\chi)$ is holomorphic for $\Re s > 1/2$. Therefore, by Claim 2 $\phi(s,\chi)$ extends meromorphically to $\Re s > 1/2$ with poles only at the zeros of $L(s,\chi)$ for $\chi \neq \chi_0$, and $\phi(s,\chi_0)$ extends meromorphically to $\Re s > 1/2$ with poles only at $s = 1$ and the zeros of $L(s,\chi_0)$. Summing up the above equality over all χ we get

$$-\frac{\mathcal{L}'(s)}{\mathcal{L}(s)} = -\frac{L'(s,\chi)}{L(s,\chi)} = \Phi_q(s) + h(s)$$

for some holomorphic function $h(s)$ in $\Re s > 1/2$. Recall that the residue of the logarithmic derivative of a function f at a pole is equal to the order of the zero of f at this point. Therefore we have:

$$\begin{aligned}
\operatorname{res}_{s=1}(\Phi_q(s)) &= \lim_{\varepsilon\to 0} \varepsilon\Phi_q(1+\varepsilon) = 1,\\
\operatorname{res}_{s=1\pm i\alpha}(\Phi_q(s)) &= \lim_{\varepsilon\to 0} \varepsilon\Phi_q(1+\varepsilon\pm i\alpha) = -\mu,\\
\operatorname{res}_{s=1\pm 2i\alpha}(\Phi_q(s)) &= \lim_{\varepsilon\to 0} \varepsilon\Phi_q(1+\varepsilon\pm 2i\alpha) = -\nu.
\end{aligned}$$

Let us now sum the values of Φ_q at these 5 points with binomial coefficients $1, 4, 6, 4, 1$. We obtain

$$\begin{aligned}
\sum_{r=-2}^{2} \binom{2+r}{4} \Phi_q(1+\varepsilon+ri\alpha) &= \sum_p \sum_\chi \frac{\chi(p)\log p}{p^{1+\varepsilon}} \left(p^{i\alpha/2} + p^{-i\alpha/2}\right)^4\\
&= \sum_{p\equiv 1\,(q)} \frac{\phi(q)\log p}{p^{1+\varepsilon}} \left(p^{i\alpha/2} + p^{-i\alpha/2}\right)^4 \geq 0.
\end{aligned}$$

On the other hand ε times the left hand side of the equation approaches $-\nu-4\mu+6-4\mu-\nu$, as $\varepsilon\to 0$. Thus $6-8\mu-2\nu\geq 0$, which implies $\mu = 0$. Therefore, $\mathcal{L}(s)$ and, hence, $L(s,\chi)$ for each χ has no zeros on $\Re s = 1$. Note also that $L(s,\chi)$ has no zeros for $\Re s > 1$ by Claim 1. □

Claim 5. $\Phi_{q,a}(s) - \frac{1}{s-1}$ is holomorphic for $\Re s \geq 1$.

Proof. Indeed, by definition

$$\Phi_{q,a}(s) = \sum_\chi \overline{\chi(a)}\phi(s,\chi) = \sum_{\chi\neq\chi_0} \overline{\chi(a)}\phi(s,\chi) + \phi(s,\chi_0).$$

From (2.1) we see that each $\phi(s,\chi)$ is holomorphic for $\Re s \geq 1$ since $L(s,\chi)$ is holomorphic and has no zeros in $\Re s \geq 1$ by Claim 2 and Claim 4. Also $\phi(s,\chi_0) - \frac{1}{s-1}$ is holomorphic for $\Re s \geq 1$ since $L(s,\chi_0)$ has a simple pole at $s=1$ and no zeros in $\Re s \geq 1$. The statement now follows. □

Claim 6. $\int_1^\infty \frac{\theta_q(x)-x}{x^2}\,dx$ converges.

Proof. Let $\bar{a}$ be an inverse of a mod q, i.e. such that $\bar{a}a \equiv 1\,(q)$. Then $\overline{\chi}(a) = \chi(\bar{a})$ and by definition

$$\begin{aligned}
\Phi_{q,a}(s) &= \sum_\chi \overline{\chi(a)} \sum_p \frac{\chi(p)\log p}{p^s} = \sum_p \Big(\sum_\chi \chi(\bar{a})\chi(p)\Big) \frac{\log p}{p^s}\\
&= \sum_{\bar{a}p\equiv 1\,(q)} \frac{\phi(q)\log p}{p^s} = \sum_{p\equiv a\,(q)} \frac{\phi(q)\log p}{p^s}.
\end{aligned}$$

Note that $\theta_q(x)$ has jumps of height $\phi(q)\log p$ at points $x = p$, where $p \equiv a\,(q)$. Thus we can write the above sum as an Riemann-Stieltjes integral and apply integration by parts:

$$= \int_1^\infty \frac{d\theta_q(x)}{x^s} = \left.\frac{\theta_q(x)}{x^s}\right|_1^\infty + s\int_1^\infty \frac{\theta_q(x)}{x^{s+1}}\,dx.$$

Replacing $x = e^t$ we finally get

$$\Phi_{q,a}(s) = s \int_0^\infty e^{-st}\theta_q(e^t)\,dt. \tag{2.2}$$

Now let $f(t) = \theta_q(e^t)e^{-t} - 1$. The function f is bounded since $\theta_q(x) = O(x)$ by Claim 3 and locally integrable since it has a discrete set of points of discontinuity. Moreover,

$$\begin{aligned} g(z) &= \int_0^\infty f(t)e^{-zt}\,dt = \int_0^\infty \theta_q(e^t)e^{-(z+1)t}\,dt - \int_0^\infty e^{-zt}\,dt \\ &= \frac{\Phi_{q,a}(z+1)}{z+1} - \frac{1}{z}, \end{aligned}$$

where the last equality follows from (2.2). Therefore by Claim 5 the function $g(z)$ extends holomorphically to $\Re s \geq 0$. We now are under the conditions of the Analytic Theorem:

Analytic Theorem. [Z] Let $f(t)$, $t \geq 0$ be a bounded and locally integrable function and suppose that the function $g(z) = \int_0^\infty f(t)e^{-zt}\,dt$, where $\Re z > 0$, extends holomorphically to $\Re z \geq 0$. Then $\int_0^\infty f(t)\,dt$ exists and equals $g(0)$.

It remains to show that

$$g(0) = \int_0^\infty (e^{-1}\theta_q(e^t) - 1)\,dt = \int_0^\infty \frac{\theta_q(e^t) - e^t}{e^t}\,dt = \int_1^\infty \frac{\theta_q(x) - x}{x^2}\,dx.$$

□

Claim 7. $\theta_q(x) \sim x,\ x \to \infty$.

Proof. This follows directly from Claim 6. □

Claim 8.

$$\pi(x,q) \sim \frac{1}{\phi(q)}\frac{x}{\log x},\ x \to \infty.$$

Proof. We have

$$\theta_q(x) = \phi(q) \sum_{\substack{p \leq x \\ p \equiv a\,(q)}} \log p \leq \phi(q) \sum_{\substack{p \leq x \\ p \equiv a\,(q)}} \log x = \phi(q)\pi(x,q)\log x.$$

Fix any $\varepsilon > 0$. Then

$$\theta_q(x) \geq \phi(q) \sum_{\substack{x^{1-\varepsilon} \leq p \leq x \\ p \equiv a\,(q)}} \log p \geq \phi(q) \sum_{\substack{x^{1-\varepsilon} \leq p \leq x \\ p \equiv a\,(q)}} (1-\varepsilon)\log x = \phi(q)(1-\varepsilon)\log x\big(\pi(x,q) + O(x^{1-\varepsilon})\big),$$

since $\pi(x,q) = O(x)$, clearly. It remains to let $\varepsilon \to 0$. □

References

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Department of Mathematics, University of Toronto, Toronto, ON Canada
E-mail address: isoprou@math.toronto.edu