Symbols and residues on surfaces

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1 Introduction

1. The classical Vieta formulae give the product and sum of roots of a polynomial equation $P(x) = 0$. Not long ago A.G. Khovanskii found multidimensional generalizations of Vieta's formulae. He considered a system of equations

$$P_1(x) = \cdots = P_n(x) = 0, \qquad x \in (\mathbb{C}^*)^n, \tag{1.1}$$

whose Newton polyhedra have sufficiently general relative locations.

In [Kh] A.G. Khovanskii computes the value of a character $\chi : (\mathbb{C}^*)^n \to \mathbb{C}^*$ at the product of all roots of the system (1.1). His formula involves numbers $[P_1, \ldots, P_n, \chi]$ that are similar to the tame symbols appeared in Parshin–Kato reciprocity laws. As it was mentioned in [Kh] this gives a motive to try to explain this result in the framework of Parshin-Kato theory.

The multidimensional generalization of the Vieta sum of roots formula is the formula for the sum of the Grothendieck residues of a rational 2-form over the roots of (1.1), discovered by O.A. Gelfond and A.G. Khovanskii (see [G-Kh]).

In the present paper we explain both results in terms of Parshin–Kato theory in the case when $n = 2$.

2. The paper is organized as follows. In Section 2 we recall the definitions of the tame symbol and the residue associated with a complete flag of subvarieties on a surface. We also formulate the Parshin 2-dimensional reciprocity laws.

Section 3 contains the main theorems (Theorem 3.2, Theorem 3.3). They state a certain reciprocity between two pairs of 1-dimensional subvarieties on a compact surface, when they are located sufficiently generally. Namely, we show that under a generality assumption the product of symbols (sum of residues) over the intersections of one pair of subvarieties is equal to the product of symbols (sum of residues) over the intersections of the other pair of subvarieties.

In Section 4 we consider a system of two equations in an open subset U of a compact algebraic surface X:

$$f(p) = g(p) = 0, \qquad p \in U, \tag{1.2}$$

where f, g are regular in U and the zero loci of f and g have no common components. Under a generality assumption we calculate the sum (product) of values of a function h over the solution points of the system (1.2) in terms of residues (symbols) at "infinite" points on $D_\infty = X \setminus U$

(Theorem 4.7, Theorem 4.6). The function h can be any rational function, regular (and non-vanishing) in U.

Section 5 is devoted to toric applications of the main theorems and the results of Section 4. We consider a system in the algebraic torus $(\mathbb{C}^*)^2$

$$P_1(x,y)=0, \quad P_2(x,y)=0, \quad x,y\in\mathbb{C}^*,$$

where P_1, P_2 are Laurent polynomials with the Newton polygons Δ_1, Δ_2. We consider the toric compactification X associated with $\Delta=\Delta_1+\Delta_2$. We calculate the symbol and the residue at "infinite points" on $D_\infty = X\setminus(\mathbb{C}^*)^2$ explicitly (Proposition 5.4, Proposition 5.9). We also obtain the Khovanskii product of roots formula and the Gelfond–Khovanskii sum of residues formulae for $n=2$.

3. I would like to thank Prof. A.G. Khovanskii for his helpful comments and constant attention to this work. This paper is based on research that will form part of my University of Toronto Ph.D thesis written under the supervision of Prof. A.G. Khovanskii.

2 Preliminaries

In this section we recall the definitions of the tame symbol and the residue on a surface and formulate the Parshin reciprocity laws.

Let X be a compact irreducible algebraic surface over an algebraically closed field k of characteristic zero. We fix a flag F on X consisting of an irreducible algebraic curve C and a point p on it.

2.1 Tame symbol

First, suppose X is regular and p is a regular point on C. Let u,t be a system of local parameters at p such that t is a local equation of C at p. Then we can define two valuations associated with the flag F. For any rational function $f\in k(X)$ let $v_C(f)$ be the order of f along C and $v_p(f)$ be the order of the restriction $ft^{-v_C(f)}|_C$ at p. For two rational function f,g put

$$v_{p,C}(f,g)=v_p(f)v_C(g)-v_p(g)v_C(f).$$

This number does not depend on the choice of local parameters and has certain geometrical meaning (see Section 4.1).

Definition 2.1. The *tame symbol* of $f,g,h\in k(X)$ at $F=\{p\in C\subset X\}$ is defined by the following formula

$$\{f,g,h\}_F=(-1)^{R(k,l,m)}[f^{v_{p,C}(g,h)}g^{v_{p,C}(h,f)}h^{v_{p,C}(f,g)}](p),$$

where $R(k,l,m)=k_1l_1m_1+k_1l_2m_1+k_1l_2m_2+k_2l_1m_1+k_2l_1m_2+k_2l_2m_1$, for $k=(k_1,k_2)=(v_C(f),v_p(f))$, $l=(l_1,l_2)=(v_C(g),v_p(g))$, and $m=(m_1,m_2)=(v_C(h),v_p(h))$.

Note that the rational function inside the square brackets has v_C and v_p being equal to zero, and hence, its value at p is a non-zero element of k.

Remark 2.2. In [Kh] A.G. Khovanskii showed that there exists a unique function of $n+1$ vectors in n-dimensional space over $\mathbb{Z}/2\mathbb{Z}$ with the same properties as the determinant, i.e., invariant under linear transformations and vanishing if the rank of the $n+1$ vectors is smaller than n. The function $R(k,l,m)$ coincides with this function for $n=2$.

Now let X be any surface (may be singular). To define the symbol associated with F we use a sequence of normalizations.

Let $\pi : \overline{X} \to X$ be the normalization of X. Let $C_{q,i}$ be a local irreducible branch of $\pi^{-1}(C)$ passing through some $q \in \pi^{-1}(p)$. Then it has a local equation t in some Zariski open set $U \subset \overline{X}$. Let $\nu : \overline{C}_{q,i} \to C_{q,i}$ be the normalization of $C_{q,i}$. Then $\nu^{-1}(q)$ has a local equation u in an open subset $V \subset \overline{C}_{q,i}$. Using u,t we define valuations $v_{C_{q,i}}(f)$ and $v_q(f)$, and the symbol $\{f,g,h\}_{F_{\text{reg}}}$ at $F_{\text{reg}} = \{q \in C_{q,i} \subset \overline{X}\}$ in the same way as in the regular case.

Now the definition of the symbol at the flag F is obtained by multiplicativity:

$$\{f,g,h\}_F = \prod_{F_{\text{reg}}} \{f,g,h\}_{F_{\text{reg}}}, \tag{2.1}$$

where the product is taken over all flags $F_{\text{reg}} = \{q \in C_{q,i} \subset \overline{X}\}$ for $q \in \pi^{-1}(p)$ and a branch $C_{q,i}$ of $\pi^{-1}(C)$ passing through q.

Further in the paper we will often say "a symbol at p along C" or just "a symbol at p" when it is clear from the context which flag is considered. We will also write $\{f,g,h\}_{p\in C}$ for the symbol at the flag $\{p \in C \subset X\}$.

The following two theorems state the reciprocity laws for the symbol on a compact algebraic surface. These theorems generalize Weil's reciprocity law for the tame symbol on a compact algebraic curve. They were first proved by A.N. Parshin (see [F-P, P]). J.-L. Brylinski and D.A. McLaughlin gave topological proof in [B-ML].

Reciprocity Law 1. *For a fixed irreducible curve C in X,*

$$\prod_{p\in C} \{f,g,h\}_{p\in C} = 1,$$

where the product is taken over all points in C and the product is finite.

Remark 2.3. Note that we can replace C by any 1-dimensional subvariety in X. In this case $\{f,g,h\}_{p\in C}$ will stand for the product of all symbols at p along the irreducible components of C passing through p.

Reciprocity Law 2. *For a fixed point p in X,*

$$\prod_{C\ni p} \{f,g,h\}_{p\in C} = 1,$$

where the product is taken over all irreducible curves C containing p and the product is finite.

2.2 Residue

We now define the residue of a rational differential 2-form $\omega \in \Omega^2(X)$ at a flag F. As before we first define the residue of ω in the case when X is regular and p is a regular point of C.

Let u, t be local parameters at p such that t is a local equation of C at p. Consider a decomposition $\omega = f(u,t)du \wedge dt$ for a rational f. Let

$$f(u,t) = \sum_{n \geq n_0} f_n(u) t^n \tag{2.2}$$

be the formal power series expansion of f in t.

Definition 2.4. The *residue* of a differential form $\omega \in \Omega^2(X)$ at $F = \{p \in C \subset X\}$ is defined to be

$$\mathrm{res}_F(\omega) = \mathrm{res}_p(f_{-1}(u)du|_C).$$

It can be shown that the residue does not depend on the choice of local parameters u, t (see [F-P]).

In general case we use the same procedure as in the definition of the symbol. Using the normalization we obtain u, t and define $\mathrm{res}_{F_{\mathrm{reg}}}(\omega)$ at $F_{\mathrm{reg}} = \{q \in C_{q,i} \subset \overline{X}\}$ as in the regular case.

Similar to (2.1) we define the residue at F by additivity:

$$\mathrm{res}_F(\omega) = \sum_{F_{\mathrm{reg}}} \mathrm{res}_{F_{\mathrm{reg}}}(\omega),$$

where the sum runs over all flags $F_{\mathrm{reg}} = \{q \in C_{q,i} \subset \overline{X}\}$ for $q \in \pi^{-1}(p)$ and $C_{q,i}$ branch of $\pi^{-1}(C)$ passing through q.

Definition 2.5. Let C be a branch of a curve on X. We say that $\omega \in \Omega^2(X)$ has a *pole* on C if in the expansion (2.2) $n_0 \leq -1$. Thus, it makes sense for ω to be "regular" on C even if $C \subset \mathrm{Sing}\, X$. In fact, in that case $\pi^*\omega$ is regular at a generic point of $\pi^{-1}(C)$.

It can readily be seen that $\mathrm{res}_{p \in C}\, \omega = 0$ unless ω has a pole on C and some other branch passing through p.

The following two theorems state the reciprocity laws that are additive versions of RL1 and RL2 (see [F-P, P]).

Reciprocity Law 3. *For a fixed irreducible curve C in X,*

$$\sum_{p \in C} \mathrm{res}_{p \in C}(\omega) = 0,$$

where the sum is taken over all points in C and the sum is finite.

Reciprocity Law 4. *For a fixed point p in X,*

$$\sum_{C \ni p} \mathrm{res}_{p \in C}(\omega) = 0,$$

where the sum is taken over all irreducible curves C containing p and the sum is finite.

3 Main theorems

Let X be a compact irreducible algebraic surface over an algebraically closed field k. Let C and D be two simple (all their components appear with coefficient 1) divisors without common components. We show that if C and D are in sufficiently general positions, relative to each other (see definition below), then there is a certain reciprocity between C and D. Namely, the product of symbols (sum of residues) at the intersection points of the components of C is equal to the product of symbols (sum of residues) at the intersection points of the components of D.

Definition 3.1. Let C, D be simple divisors without common components. Fix a decomposition $C = C_1 + C_2$, $D = D_1 + D_2$ into sum of simple divisors. We say that C and D are *expanded* with respect to the decomposition if C_1 does not intersect D_2 and C_2 does not intersect D_1.

Theorem 3.2. *Suppose C and D are expanded with respect to $C = C_1 + C_2$, $D = D_1 + D_2$. Then for three rational function $f, g, h \in k(X)$ with supports of their divisors in $C \cup D$ we have*

$$\prod_{p\in C_1\cap C_2} \{f,g,h\}^{-1}_{p\in C_1} = \prod_{p\in C_1\cap C_2} \{f,g,h\}_{p\in C_2} = \prod_{p\in D_1\cap D_2} \{f,g,h\}^{-1}_{p\in D_2} = \prod_{p\in D_1\cap D_2} \{f,g,h\}_{p\in D_1}.$$

Proof. 1. Take $p \in C_1 \cap C_2$. Since C and D are expanded there are no components of D passing through p. Thus, the divisors of f, g, h passing through p can appear only as components of C. By RL2

$$\{f,g,h\}_{p\in C_1}\{f,g,h\}_{p\in C_2} = 1.$$

and the first equality follows. The third equality is similar.

2. We will prove the second one. Consider $p \in C_2$. If there are no components of the divisors of f, g, h passing through p, except may be lying in C_2, then according to RL2,
$\{f,g,h\}_{p\in C_2} = 1$. Thus the only non-trivial symbols on C_2 appear at $p \in C_1 \cap C_2$ or
$p \in C_2 \cap D_2$ (since $C_2 \cap D_1 = \varnothing$). Applying RL1 to C_2 we get

$$1 = \prod_{p\in C_2} \{f,g,h\}_{p\in C_2} = \prod_{p\in C_1\cap C_2} \{f,g,h\}_{p\in C_2} \prod_{p\in C_2\cap D_2} \{f,g,h\}_{p\in C_2}. \tag{3.1}$$

Same arguments applied to D_2 provide

$$1 = \prod_{p\in D_2} \{f,g,h\}_{p\in D_2} = \prod_{p\in D_1\cap D_2} \{f,g,h\}_{p\in D_2} \prod_{p\in D_2\cap C_2} \{f,g,h\}_{p\in D_2}. \tag{3.2}$$

Multiplying (3.1) by (3.2) and using the same arguments as in 1 for each point of intersection $C_2 \cap D_2$ we get the required equality. □

Theorem 3.3. *Suppose C and D are expanded with respect to $C = C_1 + C_2$, $D = D_1 + D_2$. Then for any rational form $\omega \in \Omega^2(X)$ with poles (see Definition 2.5) in $C \cup D$ we have*

$$-\sum_{p\in C_1\cap C_2} \operatorname{res}_{p\in C_1} \omega = \sum_{p\in C_1\cap C_2} \operatorname{res}_{p\in C_2} \omega = -\sum_{p\in D_1\cap D_2} \operatorname{res}_{p\in D_2} \omega = \sum_{p\in D_1\cap D_2} \operatorname{res}_{p\in D_1} \omega.$$

The proof repeats the arguments of the proof of Theorem 3.2.

4 Systems of equations on X

In this section we assume that the ground field $k = \mathbb{C}$. We apply the main theorems to the following situation. Let X be a complex compact irreducible algebraic surface. Let U be an open subset of X given by eliminating a divisor "at infinity" D_∞. We set $U = X \setminus D_\infty$. Consider a system in U

$$f(p) = 0, \quad g(p) = 0, \qquad p \in U, \tag{4.1}$$

where $f, g \in k(X)$ are regular in U and the zero loci $V(f)$ and $V(g)$ have no common components. Thus, the solution set consists of a finite number of points $p \in U$ with some multiplicities $\mu(p)$ that are the local intersection numbers of $\operatorname{div}_0 f$ and $\operatorname{div}_0 g$.

First, in Section 4.1 we show that the symbol and the residue at a solution point have a nice meaning. Namely, we have

$$\{f, g, h\}_{p \in V(f)} = h(p)^{-\mu(p)},$$

for any $h \in k(X)$, regular and non-vanishing in U, and

$$\operatorname{res}_{p \in V(f)} \left(h \frac{df}{f} \wedge \frac{dg}{g} \right) = -\mu(p) h(p),$$

for any $h \in k(X)$, regular in U.

Second, if the system (4.1) is generic (see Definition 4.5) we can apply the main theorems and get formulae for the product (sum) of the values of h over the solutions of the system in terms of symbols (residues) at "infinite" points (Theorem 4.6, Theorem 4.7). These formulae are similar to Vieta's formulae when you can find the sum and the product of roots without solving the equation. In the next section we deal with a system of two polynomial equations on $(\mathbb{C}^*)^2$ where we calculate symbols (residues) at "infinity" explicitly in terms of the coefficients of the system.

4.1 Intersection numbers

Here we give a geometric meaning to the number $v_{p,C}(f, g)$ defined in Section 2.1. The corollaries are the formulae for the symbol and the residue at a solution point of the system (4.1).

Proposition 4.1. *For any two rational functions $f, g \in k(X)$ the local intersection number of their zero divisors $\operatorname{div}_0 f$ and $\operatorname{div}_0 g$ at p is equal to $-v_{p,V(f)}(f, g)$.*

Proof. By definition the local intersection number $\mu(p)$ of $\operatorname{div}_0 f$ and $\operatorname{div}_0 g$ at p is twice the leading coefficient of the Hilbert polynomial of the graded algebra

$$A = \oplus_{n=0}^{\infty} I^n / I^{n+1},$$

where I is the ideal generated by f, g in the local ring $\mathcal{O}_p$.

First we assume that the surface is regular and p is a regular point of $V(f)$. It can be easily shown that in this case A is a polynomial ring over $\mathcal{O}_p/I$ and $\mu(p) = \dim_k \mathcal{O}_p/I$.

Let u, t be local parameters at p. Since p is a regular point of $V(f)$ we may assume that $f = u^n$, where $n = v_{V(f)}(f)$. We have

$$\begin{aligned}\dim_k \frac{\mathcal{O}_p}{(f,g)} = n \dim_k \frac{\mathcal{O}_p}{(u,g)} = n \dim_k \frac{\mathcal{O}_p/(u)}{(u,g)/(u)} \\ = n \dim_k \frac{\mathcal{O}_{p,V(f)}}{(g|_{V(f)})} = n v_p(g|_{V(f)}) = -v_{p,V(f)}(f,g).\end{aligned}$$

Here $\mathcal{O}_{p,V(f)}$ is the local ring of p on $V(f)$.

Now let X be any surface. We reduce this case to the previous one using the following deformation principle.

Lemma 4.2. *Let $p \in V(f) \cap V(g)$. There is a small neighborhood B of p (in the usual topology) such that*

$$v_{p,V(f)}(f,g) = \sum_{x \in B} v_{x,V(\tilde{f})}(\tilde{f}, \tilde{g}),$$

where $\tilde{f}$ and $\tilde{g}$ are small deformations of f and g. The sum is finite since $v_{x,V(\tilde{f})}(\tilde{f}, \tilde{g}) = 0$ for all $x \in B$, but $x \in V(\tilde{f}) \cap V(\tilde{g})$.

Proof. First we show that if C is not a branch of $V(g)$ then

$$v_{p,C}(f,g) = \sum_{x \in B} v_{x,C}(f, \tilde{g}), \tag{4.2}$$

for some neighborhood B of p and small deformation $\tilde{g}$.

Let u be a local parameter on C at p. Then $g|_C = u^k \phi$, $k \geq 1$, $\phi(p) \neq 0$. Choose $\varepsilon > 0$ such that $g|_C$ has no zeroes on and inside $\|u\| = \varepsilon$ except at p. By Sard's theorem there exists ρ, $0 < \rho < \min_{\|u\|=\varepsilon} \|g|_C\|$, which is not a critical value of $\|g|_C\|$. Put $g_\tau = g|_C - \rho\tau$, $0 \leq \tau \leq 1$ and $\tilde{g} = g - \rho$. Then

$$v_p(g) = k = \frac{1}{2\pi i} \int_{\|u\|=\varepsilon} \frac{dg_\tau}{g_\tau} = \sum_{x \in \{\|u\| < \varepsilon\}} v_x(\tilde{g}),$$

since the integral is continues with respect to τ and has integer values. Let B be a neighborhood of p such that $B \cap C = \{\|u\| < \varepsilon\}$. We get

$$v_{p,C}(f,g) = -v_C(f) v_p(g) = -v_C(f) \sum_{x \in \{\|u\| < \varepsilon\}} v_x(\tilde{g}) = \sum_{x \in B} v_{x,C}(f, \tilde{g}).$$

Now, let $V(f) = C_1 \cup \cdots \cup C_s$ be the decomposition into analytic branches at p. For each C_i we choose u_i, ε_i as before and non-critical value ρ such that $0 < \rho < \min_i \min_{\|u_i\|=\varepsilon_i} \|g|_{C_i}\|$. Put $\tilde{g} = g - \rho$. Take B such that $B \cap C_i = \{\|u_i\| < \varepsilon_i\}$. Then by (4.2)

$$v_{p,V(f)}(f,g) = \sum_{i=1}^{s} v_{p,C_i}(f,g) = \sum_{i=1}^{s} \sum_{x \in B} v_{x,C_i}(f,\tilde{g}) = \sum_{x \in B} v_{x,V(f)}(f,\tilde{g})$$

It follows from RL2 that $v_{x,V(f)}(f,\tilde{g}) = v_{x,V(\tilde{g})}(\tilde{g},f)$, for each $x \in V(f) \cap V(\tilde{g})$. Repeating the same arguments as above for $f = \tilde{g}$ and $g = f$ we get

$$\sum_{x \in B} v_{x,V(\tilde{g})}(\tilde{g},f) = \sum_{x \in B} v_{x,V(\tilde{g})}(\tilde{g},\tilde{f}) = \sum_{x \in B} v_{x,V(\tilde{f})}(\tilde{f},\tilde{g}),$$

where the deformation $\tilde{f}$ is determined by all points $x \in V(\tilde{g}) \cap V(f)$. □

To complete the proof we need the similar deformation principle for the intersection number $\mu(p)$. It is provided by Samuel's formula (see [M, p.121]).

□

Corollary 4.3. *For any $h \in k(X)$, regular and non-vanishing in U*

$$\{f, g, h\}_{p \in V(f)} = h(p)^{-\mu(p)},$$

where $\mu(p)$ is the intersection number of $\operatorname{div}_0 f$ *and* $\operatorname{div}_0 g$ *at p.*

Proof. Since $h(p) \neq 0$ we have $v_C(h) = v_p(h) = 0$, for any branch C of $V(f)$, and the corollary follows from the definition of the symbol and the proposition. □

Corollary 4.4. *For any $h \in k(X)$, regular in U*

$$\operatorname{res}_{p \in V(f)} \left(h \frac{df}{f} \wedge \frac{dg}{g} \right) = -\mu(p) h(p).$$

where $\mu(p)$ is the intersection number of $\operatorname{div}_0 f$ *and* $\operatorname{div}_0 g$ *at p.*

Proof. It follows from the proposition and the general fact that

$$\operatorname{res}_{p \in C} \left(h \frac{df}{f} \wedge \frac{dg}{g} \right) = v_{p,C}(f,g) h(p),$$

for any branch C through p. Indeed, if u and t are rational functions defined in Section 2.1 we have

$$f = t^{v_C(f)} \sum_{n \geq 0} f_n(u) t^n, \qquad g = t^{v_C(g)} \sum_{n \geq 0} g_n(u) t^n.$$

Then

$$\operatorname{res}_{p\in C}\left(h\frac{df}{f}\wedge\frac{dg}{g}\right)=\operatorname{res}_{p\in C}\left(v_C(f)h\frac{dt}{t}\wedge\frac{dg_0}{g_0}\right)+\operatorname{res}_{p\in C}\left(v_C(g)h\frac{df_0}{f_0}\wedge\frac{dt}{t}\right)$$
$$=-v_C(f)\operatorname{res}_p\left(h\frac{dg_0}{g_0}\Big|_C\right)+v_C(g)\operatorname{res}_p\left(h\frac{df_0}{f_0}\Big|_C\right)=v_{p,C}(f,g)h(p).$$

□

4.2 Product and sum of roots

The classical Vieta formulae give the product and the sum of the roots of a polynomial equation. In our situation a "root" is a point in U and product and sum no longer make sense (unless U is a group, for this see Section 5). So we fix a "test" function h and find the product and the sum of the values of h over the solutions of a generic system counting multiplicities.

First, we define what we call a generic system.

Definition 4.5. Consider the system (4.1). Let C_f and C_g be the closures of $V(f)$ and $V(g)$ in X. Assume that there is no irreducible component of D_∞ which intersects both C_f and C_g. Let $D_\infty = D_1 + D_2$ be any decomposition such that D_1 contains components which intersect C_f and D_2 contains components which intersect C_g. Then $C = C_f + C_g$ and $D = D_1 + D_2$ are expanded (see Definition 3.1). In this case we say that the system is *expanded with respect to* D_∞.

Theorem 4.6. *Suppose that the system (4.1) is expanded with respect to D_∞. Then the product of the values of h over all solutions of the system counting multiplicities is given by*

$$\prod_{p\in U} h(p)^{\mu(p)} = \prod_{p\in D_1\cap D_2} \{f,g,h\}_{p\in D_1},$$

where $h\in k(X)$, regular and non-vanishing in U.

Proof. It follows from Theorem 3.2 and Corollary 4.3. □

Theorem 4.7. *Suppose that the system (4.1) is expanded with respect to D_∞. Then the sum of the values of h over all solutions of the system counting multiplicities is given by*

$$\sum_{p\in U}\mu(p)h(p) = \sum_{p\in D_1\cap D_2}\operatorname{res}_{p\in D_1}\left(h\frac{df}{f}\wedge\frac{dg}{g}\right),$$

where $h\in k(X)$, regular in U.

Proof. It follows from Theorem 3.3 and Corollary 4.4. □

5 Systems of equations on $(\mathbb{C}^*)^2$

In this section we consider the special case when U is the algebraic torus $(\mathbb{C}^*)^2$. The regular functions on $(\mathbb{C}^*)^2$ are the Laurent polynomials, i.e., the finite sums over $\mathbb{C}$ of monomials with integral exponents.

We consider a system

$$P_1(x,y)=0, \quad P_2(x,y)=0, \quad x,y\in\mathbb{C}^*, \tag{5.1}$$

where P_1, P_2 are Laurent polynomials with the Newton polygons Δ_1, Δ_2.

Let $\Delta=\Delta_1+\Delta_2$ be their Minkowski sum. We consider the toric compactification X of $(\mathbb{C}^*)^2$ associated with Δ (see [D]). The closures of the 1-dimensional orbits under the action of the torus form a divisor D_∞. If the polygons Δ_1 and Δ_2 are expanded (see Definition 5.1) the system will be expanded with respect to D_∞.

We calculate the symbols and the residues at "infinite" points explicitly in terms of the coefficients of P_1 and P_2. The product of roots and sum of residues formulae we obtain coincide with the formulae discovered by A.G. Khovanskii and O.A. Gelfond in [Kh, G-Kh]. In fact, they found these formulae for systems of n equations on $(\mathbb{C}^*)^n$ for any n. The methods they used are different from ours.

5.1 Newton polygons

We recall basic definitions from the theory of Newton polygons. Let

$$P(x,y)=\sum_{i,j}a_{ij}x^iy^j, \qquad a_{ij}\in\mathbb{C}, \quad i,j\in\mathbb{Z}$$

be a Laurent polynomial. Its *Newton polygon* is defined to be the convex hull in $\mathbb{R}^2$ of all points (i,j) such that $a_{ij}\neq 0$. Zero and one dimensional faces of a convex polygon we will call vertices and sides respectively.

Now let Δ_1, Δ_2 be two convex polygons and $\Delta=\Delta_1+\Delta_2$ their Minkowski sum. A side Γ of Δ is called *locked* if in the decomposition $\Gamma=\Gamma_1+\Gamma_2$, $\Gamma_i\in\Delta_i$ one of Γ_i is a vertex. We say that a locked side $\Gamma\in\Delta$ has *type* i if $\Gamma_i\in\Delta_i$ is a side. A vertex of Δ is called *critical* if both its adjoint sides are locked.

For each critical vertex V of Δ define the *combinatorial coefficient* k_V. It is 0 if the adjoint sides have the same type, 1 if a vector along the side of type 1 together with a vector along the side of type 2 gives positive oriented basis of $\mathbb{R}^2$, and -1 otherwise.

Definition 5.1. Two polygons Δ_1, Δ_2 are *expanded* if all sides of their Minkowski sum are locked. Geometrically it means that they do not have parallel sides with the same direction of inner normal. Thus, almost all pairs of polygons are expanded.

5.2 Toric variety associated with Δ

We recall the definition of the toric variety associated with a polygon Δ. Let $(\mathbb{R}^2)^\circ$ be the dual of $\mathbb{R}^2$ with respect to the pairing $\langle m, r\rangle = m_1r_1 + m_2r_2$. With the polygon Δ we associate a complete fan $\Sigma = \Sigma(\Delta) \subset (\mathbb{R}^2)^\circ$ of rational cones in the following way. For a cone $\sigma \subset \mathbb{R}^2$ let $\check{\sigma} \subset (\mathbb{R}^2)^\circ$ be the dual cone, $\check{\sigma} = \{r \in (\mathbb{R}^2)^\circ \mid \langle m, r\rangle \geq 0, m \in \sigma\}$. For each face $\Gamma \in \Delta$ define

$$\sigma_\Gamma = \bigcup_{r\geq 0} r\cdot(\Delta - m), \tag{5.2}$$

where m is any point strictly inside Γ. Then put $\Sigma = \{\check{\sigma}_\Gamma \mid \Gamma \in \Delta\}$.

Let $X = X_\Sigma$ be the toric variety corresponding to the fan Σ. It can be seen as a union of affine charts

$$X = \bigcup_{\check{\sigma}\in\Sigma} X_\sigma$$

glued along X_τ, for $\check{\tau} = \check{\sigma}\cap\check{\sigma'}$ a common face. The affine charts X_σ are the spectra of semigroup algebras

$$X_\sigma = \operatorname{Spec} R_\sigma, \qquad R_\sigma = \mathbb{C}[\sigma\cap\mathbb{Z}^2] = \Big\{\text{finite sums: } \sum_{m\in\sigma\cap\mathbb{Z}^2} \lambda_m z^m,\ \lambda_m \in \mathbb{C}\Big\}.$$

We use the notation $z^m = x^{m_1}y^{m_2}$, $m = (m_1, m_2) \in \mathbb{Z}^2$.

The toric variety X is a compact surface. Since $\{0\}$ is a common face of all cones in Σ we have open inclusions of the torus into the affine charts $(\mathbb{C}^*) \cong \operatorname{Spec}\mathbb{C}[\mathbb{Z}^2] \subset X_\sigma$. The torus action on itself extends to actions on the affine charts. They are compatible with the gluing and, hence, give the torus action on X. The orbits of this action are isomorphic to tori of dimension 0, 1 and 2. The closure of 2-dimensional orbit is X itself, the closures of 1-dimensional orbits form a divisor which we denote D_∞.

There is 1-1 correspondence between the faces of Δ, cones of Σ and the orbits of X:

$$\Gamma \leftrightarrow \check{\sigma}_\Gamma \leftrightarrow X_{\operatorname{lin}\sigma_\Gamma}, \quad \dim\Gamma = 2 - \dim\check{\sigma}_\Gamma = \dim X_{\operatorname{lin}\sigma_\Gamma}, \tag{5.3}$$

where $\operatorname{lin}\sigma$ denotes the maximal linear space contained in σ.

Now suppose $\Delta = \Delta_1 + \Delta_2$. The correspondence (5.3) lets the definitions of Section 5.1 be carried over to the orbits of X. Thus, the divisor D_∞ decomposes into the sum $D_\infty = D_0 + D_1 + D_2$, where D_0 is the divisor of unlocked orbit closures and D_i is the divisor of the orbit closures of type i, $i = 1, 2$. In particular, if Δ_1 and Δ_2 are expanded then $D_\infty = D_1 + D_2$.

The following proposition shows how Definition 3.1, Definition 4.5, and Definition 5.1 agree.

Proposition 5.2. *Consider the system (5.1). Let X be the toric variety associated with $\Delta = \Delta_1 + \Delta_2$. Let C_i be the closure in X of the zero locus $V(P_i)$, $i = 1, 2$. Then $C = C_1 + C_2$ and $D = D_1 + D_2$ are expanded. In particular, if Δ_1 and Δ_2 are expanded then the system is expanded with respect to D_∞.*

Proof. We will show that C_2 does not intersect D_1. Consider an affine chart X_σ in X such that the intersection $Y = X_\sigma \cap D_1$ is not empty. Then it has the form $Y = X_\tau$ for some 1-dimensional face τ of σ. The closed embedding $X_\tau \hookrightarrow X_\sigma$ is given by the map of the coordinate rings:

$$R_\sigma \to R_\tau, \qquad z^m \mapsto \chi_\tau(m) z^m,$$

where χ_τ is the characteristic function of τ.

Let $(a_1, \ldots, a_k)$ be the minimal system of generators of the semigroup $\sigma \cap \mathbb{Z}^2$. We assume that a_1 is the generator of $\tau \cap \mathbb{Z}^2$. We get a realization of X_σ as a closed subset of $\mathbb{C}^k$

$$X_\sigma \cong V(\operatorname{Ker}\phi), \quad \phi : \mathbb{C}[u_1, \ldots, u_k] \twoheadrightarrow R_\sigma,\ u_i \mapsto z^{a_i}.$$

Then the embedding $X_\tau \hookrightarrow X_\sigma$ is induced by $\mathbb{C} \to \mathbb{C}^k$, $u_1 \mapsto (u_1, 0, \ldots, 0)$, so

$$X_\tau \cong X_\sigma \cap \{u_2 = \cdots = u_k = 0\}.$$

Therefore, to find the intersections of C_2 and D_1 we write $P_2 = P'/Q'$, where P' and Q' are regular in X_σ and find zeroes of $P'|_{X_\tau} = P'(u_1, 0, \ldots, 0)$.

To get the representation $P_2 = P'/Q'$ we translate the Newton polygon of P_2. Let V be the vertex of Δ corresponding to σ and Γ be the side of Δ corresponding to τ, i.e., $\sigma = \sigma_V$ and $\tau = (\operatorname{lin}\sigma_\Gamma) \cap \sigma_V$ (see (5.2)). Since Γ is of type 1 we have

$$\Gamma = \Gamma_1 + V_2, \qquad V = V_1 + V_2, \qquad \Gamma_1, V_1 \in \Delta_1,\ V_2 \in \Delta_2. \tag{5.4}$$

Now we put $P' = z^{-V_2} P_2$. The Newton polygon Δ' of P' lies inside σ and, thus, P' is a regular function on X_σ. But because of (5.4) $\Delta' \cap \tau = \{0\}$. Thus $P'(u_1, 0, \ldots, 0) \equiv c \neq 0$. □

5.3 Product of roots

Consider the system (5.1). We want to find the product of all roots of this system counting multiplicities. This product is a point in $(\mathbb{C}^*)^2$. To locate it we fix a character $\chi : (\mathbb{C}^*)^2 \to \mathbb{C}^*$.

Let X be the toric variety associated with the sum of the Newton polygons of P_1 and P_2. According to Proposition 5.2 if the polygons are expanded the system is expanded with respect to $D_\infty = D_1 + D_2$. Applying Theorem 4.6 we get the formula for the value of the character χ at the product of the roots (Theorem 5.7). We calculate the symbols $\{P_1, P_2, \chi\}_{p \in D_1}$ explicitly in Proposition 5.4.

The following definition is introduced by A. G. Khovanskii in [Kh]. Let χ_m be a character. In coordinates x, y it is represented by a monomial $\chi_m = x^{m_1} y^{m_2}$, $m = (m_1, m_2)$.

Definition 5.3. The *symbol of P_1, P_2 and χ_m at a vertex* $V \in \Delta$ is defined by

$$[P_1, P_2, \chi_m]_V = (-1)^{R(V_1, V_2, m)} P_1(V_1)^{\det(V_2, m)} P_2(V_2)^{\det(m, V_1)},$$

where $V = V_1 + V_2$, $V_i \in \Delta_i$ is the decomposition of V, and $P_i(V_i)$ is the coefficient of the monomial corresponding to V_i in P_i, $i = 1, 2$.

Proposition 5.4. *Let p be the zero dimensional orbit corresponding to a critical vertex $V \in \Delta$ (see (5.3)). Then for a flag $\{p \in D_1 \subset X\}$ we have*

$$\{P_1, P_2, \chi_m\}_{p \in D_1} = [P_1, P_2, \chi_m]_V^{k_V},$$

where k_V is the combinatorial coefficient of V.

Proof. First suppose $k_V = 0$ and the sides adjoint to V are both of type 1. Then there are no components of the divisors of P_1, P_2 and χ_m passing through p except of D_1. Therefore, by RL2, $\{P_1, P_2, \chi_m\}_{p \in D_1} = 1$.

Now suppose $k_V \neq 0$. Let $\check{\sigma}$ be the cone corresponding to the vertex V (see (5.3)). Consider the corresponding affine chart X_σ. The intersection $Y = X_\sigma \cap D_1$ has the form X_τ for some 1-dimensional face τ of σ.

We want to find rational functions u, t such that t is a local equation of Y in some open subset of X_σ and $u|_Y$ is a local equation of p.

It is sufficient to find a function $t \in R_\sigma$ which image in the local ring $\mathcal{O}_Y$ generates the maximal ideal $\mathfrak{m}_Y$.

The local ring $\mathcal{O}_Y$ is a semigroup algebra of the semigroup $(\sigma - \tau) \cap \mathbb{Z}^2$. The maximal ideal $\mathfrak{m}_Y$ consists of functions equal to zero identically on Y, thus, is represented by the subset $\mu \cap \mathbb{Z}^2 = \big((\sigma - \tau) \setminus (\tau - \tau)\big) \cap \mathbb{Z}^2$. Clearly, any monomial z^{a_2}, where $a_2 \in \mathbb{Z}^2$ lies on the boundary of μ, generates $\mathfrak{m}_Y$. So we only have to choose a_2 to be inside σ. Let a_1 be the generator of $\tau \cap \mathbb{Z}^2$. Functions $u = z^{a_1}$ and $t = z^{a_2}$ satisfy the requirements.

Note that (a_1, a_2) form a basis of the lattice $\mathbb{Z}^2$. Let A be the unimodular transformation which sends the standard basis (e_1, e_2) to (a_1, a_2). It follows from the definition of the symbol that

$$\{P_1, P_2, \chi_m\}_{p \in Y} = [\tilde{P}_1, \tilde{P}_2, \tilde{\chi}_m]_{V \cdot A^{-1}},$$

where $\tilde{P}$ denotes the Laurent polynomial of a function P in coordinates u, t.

The following lemma is easy to check

Lemma 5.5. *For a Laurent polynomial $P(x, y)$ let $\tilde{P}(u, t)$ be its representation in coordinates u, t, where*

$$u = x^{a_{11}} y^{a_{12}}, \qquad t = x^{a_{21}} y^{a_{22}}.$$

Then

$$[\tilde{P}_1, \tilde{P}_2, \tilde{\chi}_m]_{V \cdot A^{-1}} = [P_1, P_2, \chi_m]_V^{\det A^{-1}},$$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

To compete the proof we only have to notice that the determinant $\det A^{-1} = \det A = \det(a_1, a_2)$ is exactly k_V.

□

Remark 5.6. The zero dimensional orbits $p \in D_1 \cap D_2$ and only they correspond to the critical vertices $V \in \Delta$ with $k_V \neq 0$.

Theorem 5.7. *Consider the system (5.1). Suppose the Newton polygons* Δ_1 *and* Δ_2 *are expanded. Then the value of the character* χ_m *at the product* $M(P_1, P_2)$ *of roots of the system is given by*

$$\chi_m(M(P_1, P_2)) = \prod_{V \in \Delta} [P_1, P_2, \chi_m]_V^{k_V},$$

where the product is taken over all vertices V *of the polygon* $\Delta = \Delta_1 + \Delta_2$.

Proof. It follows from Theorem 4.6, Proposition 5.2, Proposition 5.4, and Remark 5.6. □

5.4 Sum of residues

Consider the system (5.1). Let X be the toric variety associated with the sum of the Newton polygons of P_1 and P_2. As before the divisor of 1-dimensional orbit closures has the decomposition: $D_\infty = D_0 + D_1 + D_2$. For any rational 2-form $\omega \in \Omega^2(X)$ with poles in $D = D_1 + D_2$ we can apply Theorem 3.3 and get the formula for the sum of residues of ω over the roots of the system (Theorem 5.11). Its corollaries are:

(a) the generalized Euler–Jacobi formula (Corollary 5.13),

(b) formula for the sum of the values of a Laurent polynomial over the roots of a generic system on $(\mathbb{C}^*)^2$ (Corollary 5.14),

(c) formula for the sum of the residues of a polynomial 2-form over the roots of a generic polynomial system on $\mathbb{C}^2$ (Corollary 5.15).

Let P be a Laurent polynomial with Newton polygon $\Delta(P)$. Then following [G-Kh] for any Laurent polynomial Q we define the Laurent series of the rational function Q/P at a vertex V of $\Delta(P)$.

Let $P(V)$ be the coefficient in P of the monomial z^V corresponding to $V \in \Delta$. Then the constant term of the Laurent polynomial $P' = P/(P(V)z^V)$ equals 1. Thus we can write

$$\frac{1}{P'} = 1 + (1 - P') + (1 - P')^2 + \dots . \tag{5.5}$$

Note that each monomial appears only in final number of terms $(1-P')^i$ and, thus, the coefficient of each monomial of this series is well-defined.

The formal product of the series (5.5) and the Laurent polynomial $Q/(P(V)z^V)$ is called the *Laurent series of* Q/P *at the vertex* $V \in \Delta(P)$.

Definition 5.8. [G-Kh] The *residue* $\operatorname{res}_V \omega$ *at a vertex* $V \in \Delta(P)$ of 2-form $\omega = \frac{Q}{P}\left(\frac{dx}{x} \wedge \frac{dy}{y}\right)$ is the constant term of the Laurent series of Q/P at V.

Now consider the system (5.1). Let X be the toric variety associated with $\Delta = \Delta_1 + \Delta_2$. Each rational 2-form on X can be written as

$$\omega = \frac{Q}{P}\left(\frac{dx}{x} \wedge \frac{dy}{y}\right),$$

where $P = P_1 P_2$ and Q is some Laurent polynomial. Since $\Delta(P) = \Delta$ we get the definition of the residue of ω at a vertex V of Δ.

The following proposition gives the relation between two definitions of the residue.

Proposition 5.9. *Let p be the zero dimensional orbit corresponding to a critical vertex $V \in \Delta$ (see (5.3)). Then for a flag $\{p \in D_1 \subset X\}$ we have*

$$\operatorname{res}_{p\in D_1} \omega = k_V \operatorname{res}_V \omega,$$

where k_V is the combinatorial coefficient of V.

Proof. We repeat the arguments of the proof of Proposition 5.4. Let $u = z^{a_1}$, $t = z^{a_2}$ and A as before. We have

$$\omega = \frac{Q}{P}\left(\frac{dx}{x} \wedge \frac{dy}{y}\right) = \det A^{-1} \frac{\tilde{Q}}{\tilde{P}}\left(\frac{du}{u} \wedge \frac{dt}{t}\right).$$

Therefore, by the definition of the residue

$$\operatorname{res}_{p\in D_1} \omega = \det A^{-1} c_{0,0},$$

where $c_{0,0}$ is the constant term in the expansion

$$\frac{\tilde{Q}}{\tilde{P}} = \sum_{i,j} c_{i,j} u^i t^j. \tag{5.6}$$

Note that $\tilde{P} = P(V) z^{V\cdot A^{-1}}(1 + P_0)$, where P_0 is a (usual) polynomial with zero constant term. Thus, the expansion (5.6) is the Laurent series of $\tilde{Q}/\tilde{P}$ at $V \cdot A^{-1}$. Clearly, the coefficients of the Laurent series are invariant under the exponential change of variables $u = z^{a_1}$, $t = z^{a_2}$. Therefore,

$$\operatorname{res}_{p\in D_1} \omega = \det A^{-1} \operatorname{res}_{V\cdot A^{-1}} \frac{\tilde{Q}}{\tilde{P}}\left(\frac{du}{u} \wedge \frac{dt}{t}\right) = k_V \operatorname{res}_V \frac{Q}{P}\left(\frac{dx}{x} \wedge \frac{dy}{y}\right).$$

□

Now let Δ_1, Δ_2 be two polygons. We define their *extended sum* $\tilde{\Delta}$ to be the intersection of the supporting half-planes of all the unlocked sides of $\Delta = \Delta_1 + \Delta_2$. For example, if Δ_1, Δ_2 are *similar*, i.e., all sides of Δ are unlocked, then $\tilde{\Delta} = \Delta$. If Δ_1, Δ_2 are expanded then $\tilde{\Delta} = \mathbb{R}^2$.

Definition 5.10. A rational 2-form $\omega = \frac{Q}{P_1 P_2}\left(\frac{dx}{x} \wedge \frac{dy}{y}\right)$ is called *admissible* if the Newton polygon of the Laurent polynomial Q lies strictly inside the extended sum of the Newton polygons of P_1 and P_2.

The system (5.1) is called Δ*-regular* if the roots of the system "do not go to infinity". In other words, all the intersection points of the closures C_1, C_2 in X of the zero loci $V(P_1)$ and $V(P_2)$ lie in $(\mathbb{C}^*)^2$.

Theorem 5.11. *Suppose the system (5.1) is Δ-regular. Then for any admissible 2-form ω the sum of the residues of ω over the roots of the system is given by*

$$\sum_{p \in V(P_1) \cap V(P_2)} \operatorname{res}_{p \in V(P_2)} \omega = \sum_V k_V \operatorname{res}_V \omega,$$

where the last sum is taken over all critical vertices of Δ.

Proof. Let X be the toric variety associated with Δ, $D_\infty = D_0 + D_1 + D_2$ as before. Since ω is admissible it does not have poles on D_0.

Indeed, in each affine chart X_σ such that $D_0 \cap X_\sigma = X_\tau$, where τ is a 1-dimensional face of σ, choose $u = z^{a_1}$, $t = z^{a_2}$ as in the proof of Proposition 5.4. Let $\tilde{Q}$, $\tilde{P}$ be the Laurent polynomials Q, P in coordinates u, t. Then (up to sign) we can write

$$\omega = \frac{\tilde{Q}}{\tilde{P}}\left(\frac{du}{u} \wedge \frac{dt}{t}\right) = \frac{utQ'}{P'}\left(\frac{du}{u} \wedge \frac{dt}{t}\right),$$

where Q' and P' are (usual) polynomials in u, t and the constant term of P' is not zero. Therefore, ω is regular on X_τ.

By Proposition 5.2, $C = C_1 + C_2$ and $D = D_1 + D_2$ are expanded. Now the theorem follows from Theorem 3.3, Proposition 5.9 and Remark 5.6. □

Remark 5.12. It can be seen that the residue $\operatorname{res}_{p \in V(P_2)} \omega$ at $p \in V(P_1) \cap V(P_2)$ coincides with the Grothendieck residue $\left[\begin{smallmatrix} Q \\ P_1, P_2 \end{smallmatrix}\right]_p$.

Corollary 5.13. *Suppose the system (5.1) is Δ-regular. Then the sum of the residues of any form $\omega = \frac{Q}{P_1 P_2}\left(\frac{dx}{x} \wedge \frac{dy}{y}\right)$ with $\Delta(Q)$ strictly inside $\Delta = \Delta_1 + \Delta_2$ over the roots of the system is equal to zero:*

$$\sum_{p \in V(P_1) \cap V(P_2)} \operatorname{res}_{p \in V(P_2)} \omega = 0.$$

Proof. Indeed, since $\Delta(Q)$ lies strictly inside $\Delta = \Delta_1 + \Delta_2$, by the same arguments as in the proof of Theorem 5.11, ω does not have poles on D_∞. Therefore, for each point $p \in D_1 \cap D_2$ we have $k_V \operatorname{res}_V \omega = \operatorname{res}_{p \in D_1} \omega = 0$, where V is the vertex corresponding to p. □

Corollary 5.14. *Suppose the system (5.1) is Δ-regular. If Δ_1 and Δ_2 are expanded then the sum of the values of any Laurent polynomial R over the roots of the system counting multiplicities is given by*

$$\sum_{p\in V(P_1)\cap V(P_2)} \mu(p)R(p) = \sum_{V\in\Delta} k_V \operatorname{res}_V \left(R\frac{dP_1}{P_1}\wedge\frac{dP_2}{P_2}\right).$$

Proof. Since Δ_1 and Δ_2 are expanded all vertices of Δ are critical and any form ω is admissible. Thus, we can apply Theorem 5.11 to $\omega = R\frac{dP_1}{P_1}\wedge\frac{dP_2}{P_2}$ with any Laurent polynomial R. The corollary also follows from Theorem 4.7. □

We now consider a system in $\mathbb{C}^2$:

$$P_1(x,y)=0,\quad P_2(x,y)=0,\quad x,y\in\mathbb{C}, \tag{5.7}$$

where P_1, P_2 are (usual) polynomials with Newton polygons Δ_1, Δ_2. Assume that for both Δ_1 and Δ_2 the origin is a vertex and its adjoint sides lie on the coordinate axis. We say that Δ_1 and Δ_2 are *affine expanded* if all the sides of $\Delta=\Delta_1+\Delta_2$ not lying on the coordinate axis are locked.

Clearly, in this case all vertices of Δ except the origin and two vertices on the coordinate axis are critical. The extended sum of Δ_1 and Δ_2 coincides with the positive octant. Thus any polynomial 2-form $Q dx\wedge dy$ is admissible.

Corollary 5.15. *Consider the system (5.7). If Δ_1 and Δ_2 are affine expanded then for any polynomial 2-form $\omega = Q dx\wedge dy$ the sum of the residues of ω over the roots of the system is given by*

$$\sum_{p\in V(P_1)\cap V(P_2)} \operatorname{res}_{p\in V(P_2)}\omega = \sum_V k_V \operatorname{res}_V \omega,$$

where the last sum is taken over all vertices V of Δ not lying on the coordinate axis.

Proof. We consider a family of Δ-regular systems with the Newton polygons Δ_1 and Δ_2

$$P_1^t(x,y)=0,\quad P_2^t(x,y)=0,\quad x,y\in\mathbb{C},$$

that depend continuously on the parameter $0<t<1$ and the system (5.7) is the limit as $t\to 0$. Now we can apply Theorem 5.11 to each system of the family and then pass to the limit.

The corollary also follows directly from Theorem 3.3. Indeed, let X be the toric surface associated with Δ, $D_\infty = D_0+D_1+D_2$. Then ω does not have poles on D_0 and by Proposition 5.2, $C=C_1+C_2$ and $D=D_1+D_2$ are expanded. Therefore, we can apply Theorem 3.3 for $U=(\mathbb{C}^*)^2\cup D_0=\mathbb{C}^2$. □

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