

PARSHIN'S SYMBOLS AND RESIDUES, AND NEWTON
POLYHEDRA

by

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Abstract

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We introduce a new approach to the study of systems of algebraic equations whose Newton polyhedra have sufficiently general relative locations, based on the theory of tame symbols and residues due to Parshin.

We give a new explicit description of combinatorial coefficients, which are geometric invariants that reflect the relative location of a collection of n convex compact polyhedra in \mathbb{R}^n . Combinatorial coefficients are one of the main ingredients in Khovanskii's recent result on the product of the roots of a system of n algebraic equations in $(\mathbb{C}^\times)^n$ whose Newton polyhedra have sufficiently general relative locations, and in the Gelfond–Khovanskii formula for the sum of the Grothendieck residues over the roots of such systems. Our description puts the combinatorial coefficient into the framework of Parshin's theory.

We consider Parshin's theory of residues and tame symbols on toroidal varieties. It turns out to be more explicit than the general theory, and it is enriched with the combinatorics inherited from toroidal varieties. Our description of the combinatorial coefficients is essential for the proof of our main results on residues and symbols on toroidal varieties. They provide a uniform explanation of both the Khovanskii

and Gelfond–Khovanskii formulae in terms of the theory of symbols and residues on toroidal varieties, and extend them to the case of an algebraically closed field of arbitrary characteristic.

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Introduction

0.1 Overview

Let X be a smooth complex projective curve and ω a meromorphic 1-form on X . In an open neighborhood of each point $x \in X$ we can write

$$\omega = f(t) dt, \quad f(t) = \sum_{i > N} \lambda_i t^i,$$

where t is a local parameter at x . The coefficient λ_{-1} of the above series does not depend on the choice of parameter t and is called the residue of ω at x . The residue is non-zero only at the finitely many points $\Sigma \subset X$ where ω has a pole. The well-known residue formula says that the sum of the residues of ω over all points of X is zero:

$$\sum_{x \in X} \text{res}_x \omega = 0.$$

Indeed, the residue at $x \in \Sigma$ is equal to the integral of ω over any sufficiently small cycle enclosing x , divided by $2\pi i$. In the complement $X \setminus \Sigma$ the form ω is closed and the sum of the cycles is homologous to zero, thus, the residue formula follows from the Stokes theorem.

Although this proof is topological, the residue itself can be defined purely algebraically and one can give an algebraic proof of the residue formula which works over any algebraically closed field (see for example [Se], p.15).

Now let f_1, f_2 be two rational functions on X . At each point $x \in X$ consider the main terms in the Laurent expansion of f_1 and f_2 :

$$f_1 = c_1 t^{a_1} + \dots, \quad f_2 = c_2 t^{a_2} + \dots,$$

where t is a local parameter at x , and $a_i \in \mathbb{Z}$ is the order of f_i at x . The non-zero number

$$\langle f_1, f_2 \rangle_x = (-1)^{a_1 a_2} c_1^{a_2} c_2^{-a_1}$$

is called Weil's tame symbol of f_1, f_2 at x . Clearly the tame symbol is 1 unless x is in the support of the divisor of f_1 or f_2 . André Weil proved that the product of the tame symbols over all points of the curve X is equal to one ([Se], p.35):

$$\prod_{x \in X} \langle f_1, f_2 \rangle_x = 1.$$

This statement is known as Weil's reciprocity and it appears to be a multiplicative analog of the residue formula.

The tame symbol takes its origin in the classical Legendre and Hilbert symbol from the Gauss reciprocity law. One-dimensional symbols and residues were used in the work of Serre on local field theory and the theory of adèles [Se]. In higher dimensions A. Grothendieck was the first to introduce the residue map (see [Ha]). Following Serre's approach A. Parshin constructed multidimensional local field theory where he used Grothendieck's ideas to generalize the tame symbol and the residue (see [P1, P2, F-P] and also works of V. G. Lomadze [Lo]). Let X be an n -dimensional algebraic variety over an algebraically closed field k . At each complete flag of irreducible subvarieties

$$F: X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X,$$

Parshin defines the tame symbol $\langle f_1, \dots, f_{n+1} \rangle_F \in k^\times$ of $n+1$ rational functions f_1, \dots, f_{n+1} on X . Similarly, at each complete flag F he defines the residue $\text{res}_F \omega \in k$ of a rational n -form ω on X .¹ Parshin's tame symbol and residue satisfy not one but many reciprocity laws, when you fix all subvarieties X_j in the flag F except one, say X_i , and take the product (sum) over all possible irreducible subvarieties X_i in the i -th slot of the flag. We discuss this in the appendix.

¹Another approach was taken by E. Kunz and J. Lipman who considered the residue map on local cohomology groups [Ku, Li]. A. Beilinson showed in [Be] how to get Parshin's residue using a generalization of J. Tate construction [Ta]. See also works of A. Yekutieli and P. Sastry [Ye, Sa-Ye].

Surprisingly the tame symbols appeared in a recent result of the theory of Newton polyhedra. A few years ago, A. Khovanskii found the generalized Vieta formula for the product of the roots of a system of n algebraic equations. He considered a system of algebraic equations in the algebraic n -torus $(\mathbb{C}^\times)^n$

$$P_1(x) = \cdots = P_n(x) = 0, \quad x \in (\mathbb{C}^\times)^n, \quad (0.1.1)$$

where the P_i are Laurent polynomials whose Newton polyhedra Δ_i have sufficiently general relative locations.

In [Kh1] A. Khovanskii proves that the value of a character $\chi : (\mathbb{C}^\times)^n \rightarrow \mathbb{C}^\times$ at the product of the roots of the system (0.1.1) is equal to

$$\prod_{A \in \Delta} [P_1, \dots, P_n, \chi]_A^{(-1)^n c(A)},$$

where the product runs over the vertices A of the Minkowski sum $\Delta = \Delta_1 + \cdots + \Delta_n$. The definition of the number $[P_1, \dots, P_n, \chi]_A$ is similar to the one of Parshin's tame symbol. The numbers $c(A)$ are geometric invariants that reflect the combinatorics of the relative position of the polyhedra $\Delta_1, \dots, \Delta_n$ in space, and are defined topologically as a local degree of a certain real map.

Remarkably, the product of roots formula turned out to be the multiplicative analog of the formula for the sum of the Grothendieck residues of a rational n -form $\omega = \frac{Q}{P_1 \cdots P_n} dx_1 \wedge \cdots \wedge dx_n$ (Q a Laurent polynomial) over the roots of (0.1.1) due to O. Gelfond and A. Khovanskii. In [G-Kh] they showed that this sum is equal to

$$\sum_{A \in \Delta} (-1)^n c(A) \operatorname{res}_A \omega,$$

where again the sum is taken over the vertices A of $\Delta = \Delta_1 + \cdots + \Delta_n$, and $\operatorname{res}_A \omega$ is given explicitly in terms of the coefficients of the P_i and Q .

This formula was proved using similar topological arguments as in the 1-dimensional residue formula.

In [Kh1] A. Khovanskii asks: Can these two results be explained in the framework of the theory of Parshin's symbols and residues? If yes, what would be the analog of the combinatorial coefficient $c(A)$?

In the thesis we answer these questions.

1. We give an algebraic description of the combinatorial coefficient as a number of complete flags (counted with certain signs) of orbit closures on an affine toric variety, thus putting the combinatorial coefficient in the framework of Parshin's theory.

As a byproduct this gave us a new formula for the combinatorial coefficient as a number of certain flags of faces of $\Delta = \Delta_1 + \cdots + \Delta_n$, counted with signs. So far there has been no explicit combinatorial formula for the combinatorial coefficient².

2. Let X be a normal variety and D a closed codimension 1 subset of X such that in a formal neighborhood of each point the pair (X, D) is formally locally isomorphic to a pair $(X_\sigma, X_\sigma \setminus \mathbb{T})$, where X_σ is an affine toric variety and $X_\sigma \setminus \mathbb{T}$ is the closure of the codimension 1 orbits. It turns out that the tame symbol of rational functions on X with divisors in D , and the residue of a rational form on X with poles in D have a very explicit form. This, along with the above mentioned description of the combinatorial coefficient, results in a certain reciprocity law for the symbol (residue) at zero-dimensional intersections of components of D (Theorem 0.2.2).

3. This result applied to the toric compactification associated with Δ provides a uniform algebraic proof for both the Khovanskii product of roots formula and the Gelfond–Khovanskii sum of residues formula. Moreover, this proof extends these results to algebraically closed fields of arbitrary characteristic.³

Finally a few words about the notations we use. Throughout the text k is always an algebraically closed field. A variety is a reduced separated scheme of finite type over k , a subvariety is a reduced subscheme. Also \mathbb{T} denotes the algebraic n -torus

²In a special case of so-called inductively expanded collections of convex polyhedra such a formula was found by O. Gelfond in [G]. Using this formula she obtained a new formula for the mixed volume of such collections of polyhedra.

³In a series of papers [B-M1, B-M2, B-M3] J.-L. Brylinski and D. A. McLaughlin gave a topological construction for the tame symbol based on Deligne's proof of Weil's reciprocity. Recently A. Khovanskii found a new simpler topological description of the tame symbol [Kh2]. The existence of a topological description of the tame symbol provides a uniform topological explanation for the product of roots formula and the sum of residues formula.

over k , $\mathbb{T} = (k^\times)^n$, and $M = \text{Hom}_{\text{alg. gps}}(\mathbb{T}, k^\times)$ the abelian rank n group of characters of \mathbb{T} . By X_σ we always denote the affine toric variety $\text{Spec}[\sigma \cap M]$ associated with a rational convex polyhedral cone $\sigma \subset M \otimes \mathbb{R}$.

0.2 Summary of the results

Let X be a normal n -dimensional variety over an algebraically closed field k . Let D be a closed subset of X whose irreducible components are normal codimension 1 subvarieties. We say that the pair (X, D) is *toroidal* if at each point $x \in X$, (X, D, x) is formally locally isomorphic to $(X_\sigma, X_\sigma \setminus \mathbb{T}, x_0)$, where X_σ is an n -dimensional affine toric variety corresponding to a rational convex cone σ , $X_\sigma \setminus \mathbb{T}$ is the closure of codimension 1 orbits under the action of the algebraic n -torus $\mathbb{T} = (k^\times)^n$, and x_0 is a point on X_σ .

Consider the n -form $\omega_0 = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$, where (x_1, \dots, x_n) are coordinates in \mathbb{T} . This form is invariant under monomial transformations $u_i = x_1^{q_{i1}} \cdots x_n^{q_{in}}$, $Q = (q_{ij}) \in GL(n, \mathbb{Z})$ if $\det Q = 1$, and changes sign otherwise. Also a choice of coordinates in \mathbb{T} defines an orientation of the space of characters, and hence of σ . This orientation is preserved under monomial transformations if $\det Q = 1$ and changes otherwise. We call $(X_\sigma, x_0, \omega_0)$ an *equipped local model* of X at x , meaning that the form ω_0 is fixed and σ is oriented accordingly.

Definition 0.2.1. The *symbol* $[f_1, \dots, f_{n+1}]$ of $n + 1$ ordered monomials

$$f_i = c_i x^{a_i}, \quad c_i \in k^\times, \quad a_i = (a_{i1}, \dots, a_{in}) \in \mathbb{Z}^n, \quad x^{a_i} = x_1^{a_{i1}} \cdots x_n^{a_{in}}, \quad 1 \leq i \leq n + 1$$

is the non-zero element of k defined by

$$[f_1, \dots, f_{n+1}] = (-1)^B \prod_{i=1}^{n+1} c_i^{(-1)^{i+1} A_i},$$

where A_i is the determinant of the matrix obtained from $A = (a_{ij})$ by eliminating its i -th row, and $B = \sum_k \sum_{i < j} a_{ik} a_{jk} A_{ij}^k$, where A_{ij}^k is the determinant of the matrix

obtained from A by eliminating its i -th and j -th rows and its k -th column (the determinant of an empty matrix is assumed to be 1).

Invariance. The symbol is invariant under translations $x \mapsto \lambda x$. Under monomial transformations $u = x^Q$, $Q \in GL(n, \mathbb{Z})$ the symbol gets raised to the power of $\det Q$. It is also multiplicative and multiplicatively skew-symmetric in f_1, \dots, f_{n+1} (see Proposition 2.1.1).

Definition 0.2.2. Let σ be a convex rational cone in \mathbb{R}^n with apex 0. It defines an algebra A of formal power series $\sum_a \lambda_a x^a$, $a \in \sigma \cap \mathbb{Z}^n$. Consider a differential n -form $\omega = f(x) \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n}$, where

$$f(x) = x^b \sum_{a \in \sigma \cap \mathbb{Z}^n} \lambda_a x^a, \quad \lambda_a \in k, \quad x^a = x_1^{a_1} \dots x_n^{a_n}, \quad b \in \mathbb{Z}^n \quad (0.2.1)$$

is the product of a monomial and a series from A . The *toric residue* of ω is the constant term λ_{-b} of the above series.

Invariance. The residue is invariant under translations $x_i \mapsto \phi_i x_i$, where $\phi_i \in A^\times$ are invertible power series. It is also invariant under monomial transformations that preserve the form ω_0 and changes sign otherwise.

Let (X, D) be toroidal. Suppose $x \in X$ is an isolated intersection of components of D . Then in an equipped local model $(X_\sigma, x_0, \omega_0)$ at x , the point x_0 is a 0-dimensional orbit, and the cone σ has an apex. Consider a rational function f on X the support of whose divisor is in D . Then the image of f in the equipped local model $(X_\sigma, x_0, \omega_0)$ is the product of a monomial cx^a and a regular invertible function $\phi \in \widehat{\mathcal{O}}_{X_\sigma, x_0}^\times$ with $\phi(x_0) = 1$. We call this monomial the *leading monomial of f at x* . The leading monomial is defined up to monomial transformations.

Definition 0.2.3. Let (X, D) be toroidal and x an isolated point of intersection of components of D . Define the *toric symbol* $[f_1, \dots, f_{n+1}]_x$ at x of $n+1$ rational functions $f_1, \dots, f_{n+1} \in k(X)$ with the support of their divisors in D to be the symbol of the leading monomials of f_1, \dots, f_{n+1} at x .

Invariance. The toric symbol is the same for any two equipped local models that correspond to an isomorphism preserving the form ω_0 , and is reciprocal otherwise. Also it is multiplicative and skew-symmetric in f_1, \dots, f_{n+1} .

Now let ω be a rational n -form on X which is regular in $X \setminus D$. Then the image of ω in a local model at an isolated intersection x of components of D can be written as $f\omega_0$, where f is a formal power series as in (0.2.1).

Definition 0.2.4. Let (X, D) be toroidal and x an isolated point of intersection of components of D . Define the *toric residue* $\text{res}_x^{\mathbb{T}} \omega$ at x of a rational n -form ω which is regular in $X \setminus D$ to be the toric residue of its image in a local model at x .

Invariance. The toric residue is the same for any two equipped local models that correspond to an isomorphism preserving the form ω_0 , and changes sign otherwise.

The following definition comes from the theory of Newton polyhedra.

Definition 0.2.5. Let $\sigma \subset \mathbb{R}^n$ be a convex polyhedral cone with apex A . Suppose the boundary of σ is covered by n closed sets D_1, \dots, D_n , each D_i is a union of faces of σ , such that $D_1 \cap \dots \cap D_n = \{A\}$. A continuous map $f : \sigma \rightarrow \mathbb{R}^n$ is called a *characteristic map of the covering* if for every $1 \leq i \leq n$ the i -th component f_i of f is non-negative and is equal to zero precisely on D_i . All characteristic maps map σ to the positive octant \mathbb{R}_+^n such that $f^{-1}(0) = \{A\}$ and they are homotopy equivalent within this class. Fix orientations of σ and \mathbb{R}_+^n . The local degree of the germ of the restriction of a characteristic map to the boundary $\tilde{f} : (\partial\sigma, A) \rightarrow (\partial\mathbb{R}_+^n, 0)$ is called the *combinatorial coefficient of the covering* D_1, \dots, D_n of σ .

The sign of the combinatorial coefficient depends on the choice of orientations of σ and \mathbb{R}_+^n . Also it is skew-symmetric in D_1, \dots, D_n .

The following theorem provides a combinatorial description of the combinatorial coefficient.

Fix an orientation of the cone σ . Let $A = \sigma_0 \subset \dots \subset \sigma_n = \sigma$ be a complete flag of faces of σ . Let (e_1, \dots, e_n) be an ordered set of vectors such that e_i begins at σ_0

and points strictly inside σ_i . Then we say that the flag has *positive sign* if (e_1, \dots, e_n) gives positive oriented frame for σ and *negative sign* otherwise.

Theorem 0.2.1. *The combinatorial coefficient of a covering $\partial\sigma = D_1 \cup \dots \cup D_n$, $D_1 \cap \dots \cap D_n = \{A\}$ is equal to the number of all complete flags*

$$A = \sigma_0 \subset \dots \subset \sigma_{n-1} \subset \sigma,$$

counted with signs, where σ_i is a common face of exactly D_{i+1}, \dots, D_n of dimension i .

Now we will translate the definition of the combinatorial coefficient to our situation.

Definition 0.2.6. Let (X, D) be toroidal. Suppose D is a union of closed subsets D_1, \dots, D_n of pure codimension 1, such that all components of $D_1 \cap \dots \cap D_n$ are 0-dimensional. Let $x \in X$ be an isolated intersection of components of D . Then in an equipped local model $(X_\sigma, x_0, \omega_0)$ at x , the sets D_1, \dots, D_n define a covering of the boundary of the cone σ . The combinatorial coefficient of this covering of σ is called the *combinatorial coefficient of the covering D_1, \dots, D_n at x* . We denote it by $c(x)$.

Invariance. The combinatorial coefficient at x is the same for any two equipped local models at x that correspond to an automorphism of \mathbb{T} that preserves the form ω_0 , and changes sign otherwise. Also it is skew-symmetric in D_1, \dots, D_n .

As it follows from above the number

$$[f_1, \dots, f_{n+1}]_x^{c(x)}$$

no longer depends on the choice of equipped local model at x . We will call this number the *toric symbol of f_1, \dots, f_{n+1} at x associated with the covering D_1, \dots, D_n* . It is multiplicative in f_1, \dots, f_{n+1} and multiplicatively skew-symmetric in both f_1, \dots, f_{n+1} and D_1, \dots, D_n . Also there is a relation between the toric symbol associated with a covering and Parshin's tame symbol (Proposition 2.3.2).

Similarly, the number

$$c(x) \operatorname{res}_x^{\mathbb{T}} \omega$$

does not depend on the choice of equipped local model at x . We call it *toric residue of ω at x associated with the covering D_1, \dots, D_n* . It is linear in ω and skew-symmetric in D_1, \dots, D_n .

Next we formulate the main result.

Theorem 0.2.2. *Let X be a complete normal n -dimensional variety over an algebraically closed field k , and D a closed subset of X such that (X, D) is toroidal.*

Assume that

$$D = D_1 \cup \dots \cup D_n, \quad \dim(D_1 \cap \dots \cap D_n) = 0, \quad D_i = D'_i \sqcup D''_i, \quad 1 \leq i \leq n,$$

where D'_i and D''_i are disjoint pure codimension 1 subsets of D_i , $1 \leq i \leq n$. We have 2^n finite closed subsets of X :

$$S_k = E_1 \cap \dots \cap E_n, \quad E_i = D'_i \text{ or } D''_i, \quad 1 \leq i \leq n, \quad 1 \leq k \leq 2^n.$$

Then

1. (Reciprocity for the toric symbol.) Let f_1, \dots, f_{n+1} be any $n+1$ rational functions on X the supports of whose divisors are in D . Then the following 2^n numbers are equal:

$$\left(\prod_{x \in S_1} [f_1, \dots, f_{n+1}]_x^{c(x)} \right)^{(-1)^{|S_1|}} = \dots = \left(\prod_{x \in S_{2^n}} [f_1, \dots, f_{n+1}]_x^{c(x)} \right)^{(-1)^{|S_{2^n}|}},$$

where $[f_1, \dots, f_{n+1}]_x$ is the toric symbol of f_1, \dots, f_{n+1} at x , $c(x)$ is the combinatorial coefficient at x , and $|S_k|$ is the number of D''_i in the definition of S_k . If for some k , S_k is empty then the corresponding product by definition equals 1.

2. (Reciprocity for the toric residue.) Let ω be any rational n -form on X which is regular in $X \setminus D$. Then the following 2^n numbers are equal:

$$(-1)^{|S_1|} \sum_{x \in S_1} c(x) \operatorname{res}_x^{\mathbb{T}} \omega = \dots = (-1)^{|S_{2^n}|} \sum_{x \in S_{2^n}} c(x) \operatorname{res}_x^{\mathbb{T}} \omega,$$

where $\operatorname{res}_x^{\mathbb{T}} \omega$ is the toric residue at x . If for some k , S_k is empty then the corresponding sum by definition equals 0.

The proof of the theorem uses the description of the combinatorial coefficient (Theorem 0.2.1 above) and one of Parshin's reciprocity laws. In the case of the symbol, this law can be reduced to Weil's reciprocity law for a compact curve.⁴ In the case of the residue, it can be reduced to the residue formula for a compact curve.

0.3 Applications

Consider a system of n algebraic equations in the algebraic torus $\mathbb{T} = (k^\times)^n$, k algebraically closed:

$$P_1(x) = \cdots = P_n(x) = 0, \quad x \in \mathbb{T} \quad (0.3.1)$$

where the P_i are Laurent polynomials with Newton polyhedra Δ_i . We assume that neither of P_i is a monomial, hence, neither of Δ_i is a point.

Let $\Delta_1, \dots, \Delta_n$ be convex compact polyhedra in \mathbb{R}^n . Every linear functional ξ on \mathbb{R}^n defines a collection of faces $\Gamma_1^\xi, \dots, \Gamma_n^\xi$ of the polyhedra such that the restriction of ξ on Δ_i achieves its maximum precisely at Γ_i^ξ .

Definition 0.3.1. The polyhedra $\Delta_1, \dots, \Delta_n$ are called *developed* if none of them is a point and for each non-zero ξ at least one of the faces $\Gamma_1^\xi, \dots, \Gamma_n^\xi$ is a vertex.

Let Δ be the Minkowski sum of $\Delta_1, \dots, \Delta_n$. Then every face $\Gamma \subset \Delta$ has a unique decomposition as a sum of faces

$$\Gamma = \Gamma_1 + \cdots + \Gamma_n, \quad \text{where } \Gamma_i \subset \Delta_i, \quad i = 1, \dots, n. \quad (0.3.2)$$

If $\Delta_1, \dots, \Delta_n$ are developed then Δ has dimension n and in the decomposition of every proper face of Δ at least one summand is a vertex. In this case for each $A \in \Delta$ we can define the combinatorial coefficient $c(A)$ as follows.

Definition 0.3.2. Let σ_A be the cone with apex A generated by the faces of Δ that contain A . Then the boundary of σ_A is covered by the closed sets D_1, \dots, D_n , where

⁴According to A. Parshin the composition of n boundary maps from Milnor K -theory allows to represent the n -dimensional tame symbol as a certain product of 1-dimensional Weil's tame symbols. We discuss this in Appendix A.

D_i is the union of all facets of σ_A generated by those facets of Δ whose i -th summand in the decomposition (0.3.2) is a vertex. The combinatorial coefficient of this covering is called the *combinatorial coefficient* $c(A)$ of the vertex $A \in \Delta$.

If in the system (0.3.1) the polyhedra $\Delta_1, \dots, \Delta_n$ are developed then the roots of the system are isolated. Denote by $\rho(P_1, \dots, P_n)$ the product of the roots of (0.3.1) counting multiplicities. The product is a point in \mathbb{T} . To locate it we fix a character $\chi : \mathbb{T} \rightarrow k^\times$. In coordinates (x_1, \dots, x_n) it is represented by a monomial x^m , $m \in \mathbb{Z}^n$.

Definition 0.3.3. The *symbol* of P_1, \dots, P_n and a character χ at a vertex $A \in \Delta$ is the symbol of $n + 1$ monomials $[P_1(A_1)x^{A_1}, \dots, P_n(A_n)x^{A_n}, x^m]$, where $\chi = x^m$, $A = A_1 + \dots + A_n$ is the decomposition of A , and $P_i(A_i)$ is the coefficient of x^{A_i} in P_i . We denote it by $[P_1, \dots, P_n, \chi]_A$.

The following theorem was proved by A. Khovanskii in the case when $k = \mathbb{C}$ (see [Kh1]). It is a consequence of our reciprocity for the toric symbol from Theorem 0.2.2.

Theorem 0.3.1. *Suppose the Newton polyhedra $\Delta_1, \dots, \Delta_n$ of the system (0.3.1) are developed. Then the value of a character χ at the product $\rho(P_1, \dots, P_n)$ of the roots of the system is given by*

$$\chi(\rho(P_1, \dots, P_n)) = \prod_{A \in \Delta} [P_1, \dots, P_n, \chi]_A^{c(A)},$$

where the product is taken over all vertices A of the polyhedron $\Delta = \Delta_1 + \dots + \Delta_n$, and $c(A)$ is the combinatorial coefficient at A .

Now we will give an additive analog of Theorem 0.3.1.

Let P be a Laurent polynomial with Newton polyhedron $\Delta(P)$, and A a vertex of $\Delta(P)$. The constant term of the Laurent polynomial $\bar{P} = P/(P(A)x^A)$ equals 1. Thus we get a well-defined power series

$$\frac{1}{\bar{P}} = 1 + (1 - \bar{P}) + (1 - \bar{P})^2 + \dots \quad (0.3.3)$$

Definition 0.3.4. Let Q be a Laurent polynomial. The formal product of the series (0.3.3) and the Laurent polynomial $Q/(P(A)x^A)$ is called the *Laurent series of Q/P at the vertex $A \in \Delta(P)$* .

Definition 0.3.5. The *residue* $\text{res}_A \omega$ at a vertex $A \in \Delta(P)$ of a rational n -form $\omega = \frac{Q}{P} \left(\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \right)$ is the constant term of the Laurent series of Q/P at A .

Theorem 0.3.2. *Suppose the Newton polyhedra $\Delta_1, \dots, \Delta_n$ of the system (0.3.1) are developed. Then the sum of the values of a Laurent polynomial R over the roots of the system counting multiplicities is given by*

$$\sum_x \mu(x) R(x) = (-1)^n \sum_{A \in \Delta} c(A) \text{res}_A \left(\frac{x_1 \cdots x_n R}{P_1 \cdots P_n} J_P \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \right),$$

where the sum is taken over all vertices A of the polyhedron $\Delta = \Delta_1 + \cdots + \Delta_n$, J_P is the Jacobian of the polynomial map $P = (P_1, \dots, P_n)$, and $c(A)$ is the combinatorial coefficient at A .

This theorem was proved by O. Gelfond and A. Khovanskii for $k = \mathbb{C}$ (see [G-Kh]). It is a consequence of our reciprocity for toric residue from Theorem 0.2.2.

Chapter 1

Degree of polyhedral maps

In this chapter we give an explicit description of the degree of a map of polyhedral sets defined by some combinatorial data. A polyhedral set is a finite union of convex compact polyhedra intersecting in faces. Let X_1, X_2 be two polyhedral sets and $\mathcal{S}(X_1), \mathcal{S}(X_2)$ be the corresponding partially ordered sets of their faces. A continuous map $f : X_1 \rightarrow X_2$ is called polyhedral if it corresponds to some map $\phi : \mathcal{S}(X_1) \rightarrow \mathcal{S}(X_2)$ of partially ordered sets, i.e. the image of each face Γ of X_1 under f lies in the face $\phi(\Gamma)$ of X_2 (but not in a proper subface of $\phi(\Gamma)$) and the image of every subface of Γ lies in a subface of $\phi(\Gamma)$.

For any map $\phi : \mathcal{S}(X_1) \rightarrow \mathcal{S}(X_2)$ there exists a polyhedral map $f : X_1 \rightarrow X_2$ and all such maps f are homotopy equivalent in the class of polyhedral maps associated with ϕ . Thus ϕ defines a homotopy class of polyhedral maps (Proposition 1.1.1).

If X_1, X_2 are oriented and the n -th homology group of X_2 is 1-dimensional then for every map $\phi : \mathcal{S}(X_1) \rightarrow \mathcal{S}(X_2)$ the degree is defined on the group of n -cycles on X_1 . We give an explicit description of the degree in terms of the combinatorial data ϕ (Theorem 1.2.1).

One of the application of this description of the degree is a formula for the combinatorial coefficient of a collection of n convex polyhedra in \mathbb{R}^n . The combinatorial coefficients reflect the combinatorics of the relative position of the polyhedra in space. They first appeared in the result of Gelfond and Khovanskii [G-Kh] where they found

the sum of the Grothendieck residues over the zeroes of a system of algebraic equations in $(\mathbb{C} \setminus 0)^n$ whose Newton polyhedra have generic relative locations. Despite its name the combinatorial coefficient is defined topologically as a local degree of certain map. In Theorem 1.3.3 we give a purely combinatorial formula for it.

Similar construction appears in the definition of the combinatorial coefficient of a covering of the boundary of a convex polyhedral cone with an apex. The description of the combinatorial coefficient as a number of certain flags of faces of the cone (Theorem 1.3.1) is essential for the proof of our main results contained in Chapters 2 and 3.

1.1 Polyhedral maps

A polyhedral set is a finite union of convex compact polyhedra intersecting in faces. We will assume that all the polyhedra are embedded in the Euclidean space of some big dimension. Then a polyhedral set is a topological space with the topology inherited from the topology of the Euclidean space. Let X be a polyhedral set and $\mathcal{S}(X)$ be the set of all faces of all polyhedra appearing in X . The set $\mathcal{S}(X)$ is a finite partially ordered set by inclusion. A maximal chain in $\mathcal{S}(X)$ is a complete flag of faces of some polyhedron in X .

Let X_1, X_2 be two polyhedral sets and $\mathcal{S}(X_1), \mathcal{S}(X_2)$ the corresponding partially ordered sets of their faces. Consider a map $\phi : \mathcal{S}(X_1) \rightarrow \mathcal{S}(X_2)$ that preserves the partial ordering. A continuous map

$$f : X_1 \rightarrow X_2$$

is called *polyhedral associated with ϕ* if

1. the image of every face $\Gamma \in \mathcal{S}(X_1)$ lies in $\phi(\Gamma)$, but not in a proper subspace of $\phi(\Gamma)$;
2. the image of every subspace $\Gamma' \subset \Gamma$ lies in the subspace $\phi(\Gamma')$ of $\phi(\Gamma)$, but not in a proper subspace of $\phi(\Gamma')$.

For any map $\phi : \mathcal{S}(X_1) \rightarrow \mathcal{S}(X_2)$ that respects the partial ordering we can construct a polyhedral map $f : X_1 \rightarrow X_2$ associated with it. Moreover all such maps will be homotopy equivalent within the class of polyhedral sets associated with ϕ .

Proposition 1.1.1. *Each map $\phi : \mathcal{S}(X_1) \rightarrow \mathcal{S}(X_2)$ that respects the partial ordering defines a homotopy class of polyhedral maps $f : X_1 \rightarrow X_2$.*

Proof. First by a map of partially ordered sets $\phi : \mathcal{S}(X_1) \rightarrow \mathcal{S}(X_2)$ we construct a continuous piecewise linear map $f : X_1 \rightarrow X_2$ associated with ϕ .

Let us recall the construction of barycentric subdivision of a polyhedral set X . First we divide each 1-dimensional face by adding a vertex strictly inside it. Then we divide each 2-dimensional face by adding a vertex strictly inside it and connecting this vertex with all the other vertices of this face, and so on. Finally we obtain a subdivision of X into simplices. Note that there is one-to-one correspondence between the set of all simplices of the subdivision and the set of all chains in $\mathcal{S}(X)$.

Let us fix barycentric subdivisions of X_1 and X_2 . Consider a k -dimensional simplex Δ^k in the subdivision of X_1 . It corresponds to a chain $\Gamma_0 \subset \cdots \subset \Gamma_k$ in $\mathcal{S}(X_1)$. Let $\phi(\Gamma_0) \subseteq \cdots \subseteq \phi(\Gamma_k)$ be its image. It corresponds to a unique simplex in the subdivision of X_2 which we denote by $\phi(\Delta^k)$. There is a unique linear map between two simplices that maps vertices of one simplex to the prescribed vertices of the other simplex. Thus we get a map $f : X_1 \rightarrow X_2$ that sends each simplex Δ^k to the corresponding simplex $\phi(\Delta^k)$. Clearly this map agrees on the common faces of simplices of the subdivision and hence is continuous piecewise linear.

Now suppose f, g are two polyhedral maps associated with ϕ . Then for each $0 \leq t \leq 1$ the map $f_t = (1-t)f_1 + tf_2$ is also associated with ϕ . Indeed, every point x of a face Γ in X_1 is mapped to a point $f_t(x)$ on the segment joining $f_1(x)$ and $f_2(x)$. Since both $f_1(x), f_2(x)$ belong to $\phi(\Gamma)$, $f_t(x)$ also does. \square

1.2 Flags and degree of polyhedral maps

Let X be a polyhedral set. Suppose that all faces in X are oriented. In that case we say that X is *oriented*.

Let X be an oriented polyhedral set. Let F be a complete flag of faces of X , i.e. a maximal chain of elements of $\mathcal{S}(X)$, $F : \Gamma_0 \subset \cdots \subset \Gamma_n$. With the flag F we associate an ordered set of vectors (e_1, \dots, e_n) , where e_i begins at Γ_0 and points strictly inside Γ_i . Then we say that F has *positive sign* if (e_1, \dots, e_n) gives a positive oriented frame for Γ_n and *negative sign* otherwise. It is easy to check that the sign does not depend on the choice of vectors e_1, \dots, e_n .

Let $f : X_1 \rightarrow X_2$ be a continuous map between two oriented polyhedral sets X_1 and X_2 . Assume that X_2 is an *n -dimensional connected compact manifold up to codimension two*. The latter means that X_2 is a union of oriented n -dimensional convex compact polyhedra intersecting in faces such that each $(n - 1)$ -dimensional face belongs to exactly two n -dimensional faces; every two n -dimensional faces can be joined by a path of n -dimensional faces intersecting in faces of dimension $n - 1$; and the total boundary of X_2 is zero (i.e. the sum of all n -dimensional faces taken with their orientations is a cycle). Under these assumptions the degree of f is defined on the group of n -cycles on X_1 . Every n -cycle on X_1 has a form $\delta = \sum c_\alpha \Gamma_\alpha$, where Γ_α is an n -dimensional polyhedron in X_1 .

Let $f : X_1 \rightarrow X_2$ be a polyhedral map associated with $\phi : \mathcal{S}(X_1) \rightarrow \mathcal{S}(X_2)$. Let δ be an n -cycle on X_1 . For each complete flag $G : \Lambda_0 \subset \cdots \subset \Lambda_n$ of faces of X_2 define the set $\phi_\delta^{-1}(G)$ of all complete flags $F : \Gamma_0 \subset \cdots \subset \Gamma_n$ in the support of δ such that $\phi(\Gamma_i) = \Lambda_i$ for all $0 \leq i \leq n$.

Theorem 1.2.1. *Let $f : X_1 \rightarrow X_2$ be a polyhedral map associated with $\phi : \mathcal{S}(X_1) \rightarrow \mathcal{S}(X_2)$. Fix a positive complete flag $G : \Lambda_0 \subset \cdots \subset \Lambda_n$ in X_2 . Then the degree of f on an n -cycle $\delta = \sum c_\alpha \Gamma_\alpha$ is equal to the number of flags in $\phi_\delta^{-1}(G)$; each flag that appears in Γ_α is counted c_α or $-c_\alpha$ times in accordance with the sign of this flag.*

Proof. By Proposition 1.1.1 we can choose any function in the homotopy class defined

by ϕ . We take f to be the piecewise linear function constructed in the proof of Proposition 1.1.1. Then f can be viewed as a simplicial map between two simplicial complexes. Let Δ^n be any positive oriented n -dimensional simplex in X_2 . Then the value of the degree of f on $\delta = \sum c_\alpha \Gamma_\alpha$ is the number of all n -dimensional simplices in all Γ_α 's that are mapped to Δ^n ; each simplex that appears in Γ_α being counted c_α times with either sign plus if f preserves its orientation or sign minus otherwise. But the n -dimensional simplex Δ^n in X_2 defines a unique positive complete flag of faces of X_2 and every its preimage defines a unique complete flag in the support of δ , positive if f preserves its orientation and negative otherwise. \square

1.3 Combinatorial coefficient

In this section we apply Theorem 1.2.1 to get a description of the combinatorial coefficient as a number of flags of faces counted with signs.

1.3.1 Local case

Consider an n -dimensional convex polyhedral cone $\sigma \subset \mathbb{R}^n$ with apex A . We orient σ in accordance with a fixed orientation of \mathbb{R}^n .

Let D_1, \dots, D_m , $m \leq n$ be distinct non-empty closed subsets of σ , each set is the union of some facets of σ . Suppose that they cover the boundary of σ , and if $n = m$ there is no face of σ that is covered by all of them, except the vertex A :

$$\partial\sigma = D_1 \cup \dots \cup D_m, \quad \text{if } n = m \text{ then } D_1 \cap \dots \cap D_n = \{A\}. \quad (1.3.1)$$

A continuous map

$$f : \sigma \rightarrow \mathbb{R}^n$$

is called a *characteristic map* of the covering (1.3.1) if for each $1 \leq i \leq n$ the i -th component f_i of f is non-negative and vanishes precisely on those faces of σ that belong to D_i . It is easy to see that all characteristic maps map the boundary of σ

to the boundary of the positive octant \mathbb{R}_+^n such that $f^{-1}(0) \subseteq \{A\}$, and they are homotopy equivalent within the class of such maps.

Definition 1.3.1. The local degree of the germ of the restriction of a characteristic map to the boundary

$$\bar{f} : (\partial\sigma, A) \rightarrow (\partial\mathbb{R}_+^n, 0)$$

is called the *combinatorial coefficient* of the covering (1.3.1).

Clearly, the combinatorial coefficient is zero unless $m = n$. In the case when $m = n$ Theorem 1.2.1 provides us with a description of the combinatorial coefficient as the number of certain complete flags of faces of σ , counted with signs.

Denote by $\mathcal{S}(\sigma)$ the set of all faces of σ and by $\mathcal{S}(\mathbb{R}_+^n)$ the set of all faces of the positive octant \mathbb{R}_+^n . Define a map $\phi : \mathcal{S}(\sigma) \rightarrow \mathcal{S}(\mathbb{R}_+^n)$ by putting

1. $\phi(\tau) = \mathbb{R}_+^n \cap \{y_{i_1} = \cdots = y_{i_k} = 0\}$ if τ is a common face of exactly D_{i_1}, \dots, D_{i_k} ,
 $1 \leq i_l \leq n$;
2. $\phi(\sigma) = \mathbb{R}_+^n$.

It is easy to check that ϕ is a map of partially ordered sets.

As before for any complete flag $G: \gamma_0 \subset \cdots \subset \gamma_n$ of faces of \mathbb{R}_+^n define the *preimage of G under ϕ* as the set of all complete flags $\sigma_0 \subset \cdots \subset \sigma_n$ such that $\phi(\sigma_i) = \gamma_i$.

Theorem 1.3.1. *The combinatorial coefficient of a covering*

$$\partial\sigma = D_1 \cup \cdots \cup D_n, \quad D_1 \cap \cdots \cap D_n = \{A\}$$

is equal to the number of all flags counted with signs in the preimage of any complete positive flag G under ϕ .

In particular, the combinatorial coefficient is equal to the number of all complete flags

$$\sigma_0 \subset \cdots \subset \sigma_{n-1} \subset \sigma,$$

counted with signs, where σ_i is a common face of exactly D_{i+1}, \dots, D_n of dimension i .

Proof. To be able to use Theorem 1.2.1 we have to “compactify” our cones σ and \mathbb{R}_+^n . To do this we consider a pyramid Δ_A with the vertex A and the base D_0 which is a cross section of σ by a generic hyperplane. The closed subsets D_1, \dots, D_n of σ induce closed subsets of Δ_A that are unions of its faces. We will denote them by D_1, \dots, D_n as well. Consider the standard n -dimensional simplex defined in \mathbb{R}^{n+1} by $y_0 + y_1 + \dots + y_n = 1$, $y_i \geq 0$. Let Δ^n be the image of this simplex under the projection $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ to the last n coordinates y_1, \dots, y_n . We assume that the orientation of Δ^n is given by the order of y_1, \dots, y_n .

The map ϕ induces the following map (which we will also denote by ϕ) between the two sets of faces of Δ_A and Δ^n :

1. $\phi(\Gamma) = \Delta^n \cap \pi(\{y_{i_1} = \dots = y_{i_k} = 0\})$, if Γ is a common face of exactly D_{i_1}, \dots, D_{i_k} , $0 \leq i_l \leq n$,
2. $\phi(\Delta_A) = \Delta^n$.

Clearly this map is a map of partially ordered sets.

Now it remains to apply Theorem 1.2.1 for the cycle $\partial\Delta_A$ and a positive flag of faces of Δ^n containing the origin, e.g. for the flag

$$\{y_1 = \dots = y_n = 0\} \subset \{y_2 = \dots = y_n = 0\} \subset \dots \subset \{y_n = 0\} \subset \Delta^n.$$

□

Remark 1.3.1. Note that if we choose negative G then we will get minus the combinatorial coefficient. Since there are $n!$ complete flags in \mathbb{R}_+^n we get $n!$ formulae for the combinatorial coefficient. Since a choice of a complete flag G corresponds to an order of D_1, \dots, D_n we can say that the combinatorial coefficient is skew-symmetric in D_1, \dots, D_n .

1.3.2 Global case

We will recall the definition of the combinatorial coefficient for a collection of n convex compact polyhedra in \mathbb{R}^n . Then as an application of Theorem 1.2.1 we will obtain a

combinatorial formula for it.

Let $\Delta_1, \dots, \Delta_n$ be a collection of convex compact polyhedra in \mathbb{R}^n . Every linear functional ξ on \mathbb{R}^n defines a collection of faces $\Gamma_1^\xi, \dots, \Gamma_n^\xi$ of the polyhedra such that the restriction of ξ on Δ_i achieves its maximum precisely at Γ_i^ξ .

Definition 1.3.2. The polyhedra $\Delta_1, \dots, \Delta_n$ are called *developed* if none of them is a point and for each non-zero ξ at least one of the faces $\Gamma_1^\xi, \dots, \Gamma_n^\xi$ is a vertex.

Let Δ be the Minkowski sum of $\Delta_1, \dots, \Delta_n$, i.e.

$$\Delta = \{x_1 + \dots + x_n \mid x_i \in \Delta_i\} \subset \mathbb{R}^n.$$

It is not hard to see that Δ is a convex compact polyhedron as well. Every face $\Gamma \subset \Delta$ has a unique representation as a sum of faces

$$\Gamma = \Gamma_1 + \dots + \Gamma_n, \quad \text{where } \Gamma_i \subset \Delta_i, \quad i = 1, \dots, n. \quad (1.3.2)$$

The representation (1.3.2) is called the *decomposition* of the face Γ , and the face $\Gamma_i \subset \Delta_i$ is called the *i -th summand* of the decomposition.

A face $\Gamma \subset \Delta$ is called *locked* if in the decomposition of Γ at least one summand is a vertex. A vertex A of Δ is called *critical* if all the proper faces of Δ that contain A are locked.

Note that if the polyhedra $\Delta_1, \dots, \Delta_n$ are developed then the Minkowski sum Δ is n -dimensional and each vertex A of Δ is critical.

Now we will define the combinatorial coefficient at a critical vertex $A \in \Delta$. Consider a map

$$f : \Delta \rightarrow \mathbb{R}^n$$

whose i -th component f_i is non-negative and vanishes precisely on the faces $\Gamma \subset \Delta$ whose i -th summand is a vertex. Such maps are called *characteristic*. Since all proper faces of Δ that contain A are locked f maps the boundary of Δ about A to the boundary of the positive octant and $f^{-1}(0) = \{A\}$. All characteristic maps are homotopy equivalent within the class of such maps.

Definition 1.3.3. The *combinatorial coefficient* $c(A)$ of A is the local degree of the germ of the restriction of a characteristic map to the boundary

$$\bar{f} : (\partial\Delta, A) \rightarrow (\partial\mathbb{R}_+^n, 0).$$

The combinatorial coefficient depends on the orientation of \mathbb{R}^n and the order of the polyhedra $\Delta_1, \dots, \Delta_n$.

We are going to show that any characteristic map in fact corresponds to a covering of a certain cone as described in the local case (see Section 1.3.1).

Let A be a critical point of Δ . Consider the cone σ_A with apex A generated by the faces of Δ containing A . For each $1 \leq i \leq n$ define the set D_i to be the union of all facets of σ_A generated by those facets of Δ whose i -th summand is a vertex. The following simple statement shows that the sets D_i are well-defined.

Proposition 1.3.2. *Let Γ be a face of Δ whose i -th summand is a vertex. Then every face of Γ has the same property.*

Since the vertex A is critical, the boundary of σ_A is covered by the closed sets D_1, \dots, D_n , and hence the local degree of $\bar{f} : (\partial\Delta, A) \rightarrow (\partial\mathbb{R}_+^n, 0)$ is the combinatorial coefficient of the covering (see Definition 1.3.1). Therefore, we get the following formula for the combinatorial coefficient of a critical vertex.

Theorem 1.3.3. *Let $\Delta_1, \dots, \Delta_n$ be a collection of convex polyhedra in \mathbb{R}^n , Δ their Minkowski sum. Let A be a critical vertex of Δ . Denote by D_i the union of all facets Γ of Δ containing A whose i -th summand in the decomposition $\Gamma = \Gamma_1 + \dots + \Gamma_n$ is a vertex. Then the combinatorial coefficient $c(A)$ at A is equal to the number of all complete flags*

$$A = \Gamma_0 \subset \Gamma_1 \subset \dots \subset \Gamma_{n-1} \subset \Delta,$$

counted with signs, where Γ_i is a common face of exactly D_{i+1}, \dots, D_n of dimension i .

Chapter 2

Toric symbol

In this chapter we define the toric symbol and prove the reciprocity for the toric symbol on toroidal varieties.

We start with considering the case of an n -dimensional affine toric variety X_σ which has a 0-dimensional orbit under the action of the algebraic n -torus \mathbb{T} . We define the toric symbol $[f_1, \dots, f_{n+1}]$ of $n+1$ rational functions on X_σ the support of whose divisors is in $X_\sigma \setminus \mathbb{T}$. Next let (X, D) be a toroidal pair (see Section 2.3.1). Then locally in a formal neighborhood of a 0-dimensional intersection point of components of D the variety X looks like an affine toric variety with a 0-dimensional orbit. This allows us to define the toric symbol $[f_1, \dots, f_{n+1}]_x$ at such “fixed” points x of X .

Now assume that the components of D are divided into n sets, where n is the dimension of X . Thus $D = D_1 \cup \dots \cup D_n$ and assume that $D_1 \cap \dots \cap D_n$ is 0-dimensional. At each point of intersection $x \in D_1 \cap \dots \cap D_n$ we define the combinatorial coefficient $c(x)$. As we have already seen in Chapter 1 it can be described as the number of certain flags of components of D and their intersections. This gives a connection between the number $[f_1, \dots, f_{n+1}]_x^{c(x)}$ and Parshin’s tame symbol. On the other hand it is very symmetric with respect to D_1, \dots, D_n . These observations along with the multidimensional analog of Weil’s reciprocity are the essentials of the proof of our reciprocity for the toric symbol.

2.1 Symbol of monomials

Definition 2.1.1. Consider an ordered collection of $n + 1$ monomials in n variables with coefficients in a field k :

$$c_i x^{a_i} = c_i x_1^{a_{i1}} \dots x_n^{a_{in}}, \quad c_i \in k^\times, \quad a_i = (a_{i1}, \dots, a_{in}) \in \mathbb{Z}^n, \quad 1 \leq i \leq n + 1.$$

Let $A = (a_{ij}) \in M_{n+1, n}(\mathbb{Z})$ be the matrix whose rows are the vectors of exponents a_i . Then the *symbol of $n + 1$ monomials* is the non-zero element of k defined by

$$[c_1 x^{a_1}, \dots, c_{n+1} x^{a_{n+1}}] = (-1)^B \prod_{i=1}^{n+1} c_i^{(-1)^{i+1} A_i},$$

where A_i is the determinant of the matrix obtained from A by eliminating its i -th row, and

$$B = \sum_k \sum_{i < j} a_{ik} a_{jk} A_{ij}^k,$$

where A_{ij}^k is the determinant of the matrix obtained from A by eliminating its i -th and j -th rows and its k -th column.

Proposition 2.1.1. *Let $f_i = c_i x^{a_i}$, $1 \leq i \leq n + 1$, be monomials. The symbol has the following properties:*

1. (Multiplicativity) Suppose f_i is a product of two monomials $f_i = f'_i f''_i$. Then

$$[f_1, \dots, f'_i f''_i, \dots, f_{n+1}] = [f_1, \dots, f'_i, \dots, f_{n+1}] [f_1, \dots, f''_i, \dots, f_{n+1}];$$

2. (Multiplicative skew-symmetry)

$$[f_1, \dots, f_i, \dots, f_j, \dots, f_{n+1}] = [f_1, \dots, f_j, \dots, f_i, \dots, f_{n+1}]^{-1};$$

3. (Invariance)

(i) Let $u = x^Q$ be a monomial change of coordinates, i.e.

$$u_i = x_1^{q_{i1}} \dots x_n^{q_{in}}, \quad Q = (q_{ij}) \in GL(n, \mathbb{Z}).$$

Then

$$[\bar{f}_1, \dots, \bar{f}_{n+1}] = [f_1, \dots, f_{n+1}]^{\det Q},$$

where $\bar{f}_i = c_i u^{a_i} = c_i x^{a_i Q}$ and $f_i = c_i x^{a_i}$.

(ii) Let $y = \lambda x$ be a translation, i.e. $y_i = \lambda_i x_i$, $\lambda_i \in k^\times$, $1 \leq i \leq n$. Then

$$[f'_1, \dots, f'_{n+1}] = [f_1, \dots, f_{n+1}],$$

where $f'_i = c_i y^{a_i} = c_i \lambda^{a_i} x^{a_i}$ and $f_i = c_i x^{a_i}$.

Proof. Modulo the sign $(-1)^B$ all the properties follow easily from the properties of the determinant.

To take care of the sign we give an invariant description of B , following [Kh1]. Consider B as a $\mathbb{Z}/2\mathbb{Z}$ -valued function of the rows a_1, \dots, a_{n+1} of the matrix A . It is easy to see that $B = B(a_1, \dots, a_{n+1})$ is multilinear and its value on each collection of $n+1$ standard vectors $(e_{i_1}, \dots, e_{i_{n+1}})$ is 0 if more than two of the vectors $e_{i_1}, \dots, e_{i_{n+1}}$ coincide; and 1 otherwise.

Now define a function $B' = B'(a_1, \dots, a_{n+1})$ to be 0 if the rank of (a_1, \dots, a_{n+1}) is less than n ; and $\lambda_1 + \dots + \lambda_{n+1} + 1 \pmod{2}$ if the vectors a_1, \dots, a_{n+1} satisfy a (unique) non-trivial relation $\lambda_1 a_1 + \dots + \lambda_{n+1} a_{n+1} = 0$. The function B' is multilinear and on each collection $(e_{i_1}, \dots, e_{i_{n+1}})$ the functions B' and B take the same value. Therefore $B = B'$, in particular, B is symmetric and invariant under non-degenerate transformations. \square

2.2 Toric symbol on affine toric varieties

In this section we consider an affine toric variety X_σ over an algebraically closed field k , associated with a convex rational polyhedral cone σ in \mathbb{R}^n . We will assume that σ has apex 0, so X_σ has a fixed orbit of the action of the torus.

We will define the toric symbol $[f_1, \dots, f_{n+1}]$ of $n+1$ rational functions f_1, \dots, f_{n+1} on X_σ , with supports of their divisors in $X_\sigma \setminus \mathbb{T}$. To do this for each rational function

we define its leading monomial and then use the definition of the previous section. As we will show the toric symbol almost coincides with Parshin's tame symbol at a complete flag of orbit closures on X_σ (see Proposition 2.2.1).

We will start with necessary definitions from the theory of toric varieties. Let $\mathbb{T} = (k^\times)^n$ be the n -dimensional algebraic torus over an algebraically closed field k . Let $M = \text{Hom}_{\text{alg.gps}}(\mathbb{T}, k^\times)$ be the group of characters of \mathbb{T} . It is an abelian group of rank n . Denote $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ the corresponding real n -dimensional vector space. The choice of coordinates in \mathbb{T} defines isomorphisms $M \cong \mathbb{Z}^n$ and $M_{\mathbb{R}} \cong \mathbb{R}^n$. Automorphisms of \mathbb{T} correspond to monomial changes of coordinates

$$u_i = x^{q_i} = x_1^{q_{i1}} \dots x_n^{q_{in}}, \quad Q = (q_{ij}) \in GL(n, \mathbb{Z}). \quad (2.2.1)$$

Let σ be a convex rational polyhedral cone of dimension n in $M_{\mathbb{R}}$. Then it defines an affine toric variety

$$X_\sigma = \text{Spec } k[\sigma \cap M],$$

where $k[\sigma \cap M]$ is the semigroup algebra of the semigroup $\sigma \cap M$. It contains \mathbb{T} as a dense open subset $\mathbb{T} = \text{Spec } k[M] \hookrightarrow X_\sigma$. The action of \mathbb{T} on itself extends to the action of \mathbb{T} on X_σ . The orbits of the action are the tori of closed subsets $X_\tau \subset X_\sigma$, where τ is a face of σ . We will assume that σ has an apex, so X_σ has a closed orbit x_0 .

Consider a form

$$\omega_0 = \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n},$$

where x_1, \dots, x_n are coordinates in \mathbb{T} . Note that under monomial changes of coordinates (2.2.1) the form ω_0 is preserved if $\det Q = 1$ and changes sign if $\det Q = -1$. Therefore, ω_0 provides an analog of orientation on X_σ .

Furthermore, the choice of coordinates in \mathbb{T} defines an orientation of $M_{\mathbb{R}} = \mathbb{R}^n$. A monomial change of coordinates (2.2.1) defines an isomorphism $R_Q : M_{\mathbb{R}} \rightarrow M_{\mathbb{R}}$, where R_Q is the right multiplication by the matrix Q . It preserves the orientation of $M_{\mathbb{R}}$ if and only if $\det Q = 1$. Therefore, the orientation of $M_{\mathbb{R}}$ and hence of σ is uniquely determined by the form ω_0 .

Definition 2.2.1. We will write $(X_\sigma, x_0, \omega_0)$ meaning that X_σ is an affine toric variety corresponding to a cone σ with an apex, with closed orbit x_0 , and fixed form ω_0 . We also assume that the cone σ has orientation defined by ω_0 . We will call $(X_\sigma, x_0, \omega_0)$ an *equipped toric variety with closed orbit*.

Now we pass to the definition of the toric symbol on affine toric varieties.

Let $\widehat{\mathcal{O}}_{X_\sigma, x_0}$ be the completion of the local ring of x_0 on X_σ , K its field of fractions. Consider $f \in K$, the support of whose divisor lies in $X_\sigma \setminus \mathbb{T}$. Let x_1, \dots, x_n be coordinates in \mathbb{T} . Then we can write

$$f = cx^a \phi, \quad c \in k^\times, \quad x^a = x_1^{a_1} \dots x_n^{a_n}, \quad a_i \in \mathbb{Z},$$

where $\phi \in \widehat{\mathcal{O}}_{X_\sigma, x_0}^\times$ satisfies $\phi(x_0) = 1$.

Definition 2.2.2. The monomial cx^a above is called the *leading monomial* of f .

Now we can define the toric symbol of $n + 1$ functions $f_1, \dots, f_{n+1} \in K$, the supports of whose divisors lie in $X_\sigma \setminus \mathbb{T}$, as the symbol of their leading monomials. Although the leading monomial depends on the choice of coordinates in \mathbb{T} , it follows from Proposition 2.1.1 that the toric symbol is well-defined on the equipped toric variety $(X_\sigma, x_0, \omega_0)$.

Definition 2.2.3. Let $(X_\sigma, x_0, \omega_0)$ be an equipped toric variety with closed orbit x_0 . Let $f_1, \dots, f_{n+1} \in K$ with the support of their divisors in $X_\sigma \setminus \mathbb{T}$. Then their *toric symbol* $[f_1, \dots, f_{n+1}]$ is the symbol of the their leading monomials.

By Proposition 2.1.1 the toric symbol is multiplicative and multiplicatively skew-symmetric.

As it follows from Proposition 2.1.1 the toric symbol is reciprocal under coordinate changes in \mathbb{T} that change the sign of ω_0 . On the other hand Parshin's symbol is defined intrinsically without referring to coordinates (see Appendix A). The relation between Parshin's symbol and the toric symbol is given in the following proposition.

Let us recall one definition from Chapter 1. Fix the orientation of the cone σ given by the choice of ω_0 . Let F_σ be a complete flag of faces of σ :

$$F_\sigma : \quad 0 = \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_{n-1} \subset \sigma_n = \sigma.$$

Let (e_1, \dots, e_n) be an ordered set of vectors such that e_i begins at σ_0 and points strictly inside σ_i . Then we say that F_σ has *positive sign* if (e_1, \dots, e_n) gives a positive oriented frame for σ and *negative sign* otherwise.

Now let F be a complete flag of orbit closures on X_σ corresponding to a complete flag F_σ of faces of σ . We say that F has *positive (negative) sign* in accordance with the sign of F_σ . We put $\text{sgn } F = 1$ ($\text{sgn } F = -1$) if F has positive (negative) sign.

Proposition 2.2.1. *Let $(X_\sigma, x_0, \omega_0)$ be an equipped toric variety with closed orbit x_0 . Then for any $n + 1$ functions $f_1, \dots, f_{n+1} \in K$ with the support of their divisors in $X_\sigma \setminus \mathbb{T}$ we have*

$$\langle f_1, \dots, f_{n+1} \rangle_F = [f_1, \dots, f_{n+1}]^{\text{sgn } F},$$

where $\langle f_1, \dots, f_{n+1} \rangle_F$ is Parshin's symbol at a complete flag F of orbit closures and $\text{sgn } F$ is the sign of F .

Proof. Let F be a complete flag of orbit closures in X_σ .

$$F : \quad x_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

and let F_σ be the corresponding flag of faces of σ :

$$F_\sigma : \quad 0 = \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_{n-1} \subset \sigma_n = \sigma.$$

Let x_1, \dots, x_n be coordinates in \mathbb{T} , $M \cong \mathbb{Z}^n$. Inside each σ_i , $1 \leq i \leq n$ choose a lattice point $q_i \in \mathbb{Z}^n$ at the unit integer distance from σ_{i-1} . Let $u_i = x^{q_i}$ be a monomial change of coordinates in \mathbb{T} . Then the rational functions $u_i = x^{q_i}$, $1 \leq i \leq n$ give a system of local parameters corresponding to F (see Appendix A). Therefore by Appendix A, Remark A.1.1, and by Proposition 2.1.1

$$\langle f_1, \dots, f_{n+1} \rangle_F = [c_1 u^{k_1}, \dots, c_{n+1} u^{k_{n+1}}] = [c_1 x^{k_1}, \dots, c_{n+1} x^{k_{n+1}}]^{\det Q},$$

where $Q = (q_1, \dots, q_n)$ and $c_i x^{k_i}$ is the leading monomial of f_i . It remains to note that $\det Q = \operatorname{sgn} F$.

□

2.3 Reciprocity for the toric symbol on toroidal varieties

In this section we consider varieties that locally at each point look like an affine toric variety. We will define the toric symbol at “fixed” points in a similar way as we did for toric varieties, and prove one of our main results, the reciprocity for the toric symbol.

2.3.1 Toroidal pair

We recall the definition of a toroidal pair. Let X be a normal n -dimensional variety over an algebraically closed field k . Let D be a closed subset of X every irreducible component of which is a codimension 1 normal subvariety of X . We say that the pair (X, D) is *toroidal*¹ if for every closed point $x \in X$ there exists an n -dimensional torus \mathbb{T} , an affine toric variety X_σ , corresponding to a rational convex n -dimensional cone σ , and a point x_0 in X_σ , such that (X, D, x) is formally locally isomorphic to $(X_\sigma, X_\sigma \setminus \mathbb{T}, x_0)$. The latter means the isomorphism of the formal completions of the local rings

$$\widehat{\mathcal{O}}_{X,x} \cong \widehat{\mathcal{O}}_{X_\sigma, x_0},$$

such that the image of the ideal of D is mapped to the image of the ideal of $X_\sigma \setminus \mathbb{T}$. We call (X_σ, x_0) a *local model* of (X, D) at x . As before consider the n -form

$$\omega_0 = \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n},$$

¹In [TE] such pair (X, D) is called a toroidal embedding without self-intersections.

where (x_1, \dots, x_n) are coordinates in \mathbb{T} . We call $(X_\sigma, x_0, \omega_0)$ an *equipped local model* of X at x .

Let $D = \bigcup_{i \in I} E_i$ be the decomposition of D into irreducible components. The components of the sets $\bigcap_{i \in J} E_i \setminus \bigcup_{i \notin J} E_i$, where $J \subset I$, are non-singular and define a stratification of X ([TE], p. 57). In particular $X \setminus D$ is non-singular. The components of $\bigcap_{i \in J} E_i$ are normal and are the closures of the strata. Furthermore, for each $x \in X$ the closures of strata which contain x correspond formally to the closures of the orbits in a local model (X_σ, x_0) at x .

We denote by $\text{St}_i(X)$ the set of all i -dimensional strata, and by $\overline{\text{St}}_i(X)$ the set of the closures of the i -dimensional strata. Note that if $x \in \text{St}_0(X)$ then in every local model (X_σ, x_0) at x the cone σ has an apex and x_0 is the closed orbit in X_σ .

Before we define the toric symbol, let us describe what coordinate transformations relate different local models at a point $x \in X$.

Proposition 2.3.1. *Let (X, D) be a toroidal pair, $x \in \text{St}_0(X)$. Then for any two local models (X_σ, x_0) and $(X_{\sigma'}, x'_0)$ at x , every isomorphism*

$$\pi : \widehat{\mathcal{O}}_{X_\sigma, x_0} \cong \widehat{\mathcal{O}}_{X_{\sigma'}, x'_0}$$

that maps the image of the ideal of $X_\sigma \setminus \mathbb{T}$ to the image of the ideal of $X_{\sigma'} \setminus \mathbb{T}'$, is induced by a change of coordinates of the form

$$y_i = \phi_i x_1^{q_{i1}} \dots x_n^{q_{in}}, \quad \phi_i \in \widehat{\mathcal{O}}_{X_\sigma, x_0}^\times, \quad Q = (q_{ij}) \in GL(n, \mathbb{Z}),$$

where x_1, \dots, x_n and y_1, \dots, y_n are coordinates in the tori \mathbb{T} and \mathbb{T}' , respectively.

Proof. Let $\Sigma(x)$ be the union of all strata Z whose closure \overline{Z} contains x . Define the group $M(x)$ of the Cartier divisors on $\Sigma(x)$, supported on $\Sigma(x) \cap D$ and the subsemigroup $M(x)_+$ of effective divisors. For each local model (X_σ, x_0) at x , $M(x)$ is canonically isomorphic to the group of characters M of X_σ , and $M(x)_+$ is canonically isomorphic to the semigroup $\sigma \cap M$ ([TE], p. 61). Therefore the semigroups $\sigma \cap M$ and $\sigma' \cap M'$ are isomorphic. In a system of coordinates x_1, \dots, x_n and y_1, \dots, y_n it corresponds to a monomial transformation $y = x^Q$, $Q \in GL(n, \mathbb{Z})$.

To describe all isomorphisms $\pi : \widehat{\mathcal{O}}_{X_\sigma, x_0} \cong \widehat{\mathcal{O}}_{X_{\sigma'}, x'_0}$ it suffices to describe all automorphisms α of $\widehat{\mathcal{O}}_{X_\sigma, x_0}$ that fix the orbits of X_σ . Let x_1, \dots, x_n be coordinates in \mathbb{T} . Then the ring $\widehat{\mathcal{O}}_{X_\sigma, x_0}$ can be identified with the ring of all formal power series in x_1, \dots, x_n whose Newton diagram lies in $\sigma \cap M$, where M is identified with \mathbb{Z}^n . Denote this ring by A . Let S be a multiplicative subset of A consisting of all elements ϕx^a , where $a \in \sigma \cap M$ and ϕ is an invertible element of A . Then for every automorphism α , $\alpha(S) \subseteq S$. Indeed, since α fixes the orbits of X_σ , it maps every ideal (x^a) to itself. Thus $\alpha(x^a) = \phi x^a$, for some invertible ϕ . Therefore, α induces an automorphism α_S of the localization A_S . Note that $x_1, \dots, x_n \in A_S$, since the elements of $\sigma \cap M$ generate M as a group. Therefore, for each $1 \leq i \leq n$, $\alpha_S(x_i) = \phi_i x_i$, for some invertible ϕ_i .

Conversely, every map $x_i \mapsto \phi_i x_i$, $1 \leq i \leq n$, $\phi_i \in A^\times$ defines an automorphism α of A , which fixes the orbits. Indeed, for every element $f \in A$, $f = \sum_a \lambda_a x^a$, $a \in \sigma \cap M$, put

$$\alpha(f) = \sum_{a \in \sigma \cap M} \lambda_a \phi^a x^a, \quad \phi^a = \phi_1^{a_1} \dots \phi_n^{a_n}. \quad (2.3.1)$$

This is a well-defined power series. To see this note that since σ has an apex it lies strictly inside a half-space supported by some hyperplane. Each monomial has its height with respect to the hyperplane and there are only finitely many monomials whose height is no greater than a fixed number. Therefore, the coefficient of each monomial x^b of the series (2.3.1) is defined by finitely many series $\lambda_a \phi^a x^a$, where the height of a is at most the height of b .

Since all the monomials in the series (2.3.1) belong to the semigroup $\sigma \cap M$, the series define an element of A . It is easy to check that α is in fact a homomorphism. Also it is clearly invertible. \square

2.3.2 Toric symbol

Definition 2.3.1. Let f_1, \dots, f_{n+1} be rational functions on X with support of their divisors in D . Define the *toric symbol* $[f_1, \dots, f_{n+1}]_x$ at a point $x \in \text{St}_0(X)$ to be

the toric symbol of the images of f_1, \dots, f_{n+1} in an equipped local model $(X_\sigma, x_0, \omega_0)$ at x .

Remark 2.3.1. Invariance. Let $(X_{\sigma'}, x'_0, \omega'_0)$ and $(X_{\sigma''}, x''_0, \omega''_0)$ be two equipped local models at x . Let $\pi : \widehat{\mathcal{O}}_{X_{\sigma'}, x'_0} \cong \widehat{\mathcal{O}}_{X_{\sigma''}, x''_0}$ be an isomorphism of the corresponding formal completions of the local rings. For a rational function f on X the support of whose divisor lies in D , consider its images f' and f'' in the two equipped local models. Then according to Proposition 2.3.1 the leading monomials of f' and f'' are related by a composition of a monomial transformation and a translation: $x_i \mapsto \lambda_i x^{q_i}$, $\lambda_i = \phi_i(x'_0)$. Therefore, by Proposition 2.1.1 the toric symbol is the same for the two equipped local models if $\pi(\omega'_0) = \omega''_0$, and is reciprocal otherwise. Also it is multiplicative and multiplicatively skew-symmetric in f_1, \dots, f_{n+1} . Note that the invariance of the choice of local model also follows from Proposition 2.2.1.

2.3.3 Toric symbol and covering

Definition 2.3.2. Let (X, D) be a toroidal pair. We say that (D_1, \dots, D_n) is a *reasonable covering of D* if

$$D = D_1 \cup \dots \cup D_n,$$

where each D_i is the union of some irreducible components of D , and

$$D_1 \cap \dots \cap D_n \subseteq \text{St}_0(X).$$

Let (D_1, \dots, D_n) be a reasonable covering of D . Then at each $x \in \text{St}_0(X)$ we get a covering of the boundary of the cone σ with apex A in an equipped local model $(X_\sigma, x_0, \omega_0)$ at x . Indeed, each facet of σ corresponds to the closure of a codimension 1 orbit in X_σ , thus, to a component of D containing x (see Section 2.3.1). Therefore the boundary of σ is covered by closed subsets $D_{i_1}^\sigma, \dots, D_{i_m}^\sigma$, each $D_{i_k}^\sigma$ being the union of all facets that correspond to components of D lying in D_{i_k} . Also, if $m = n$ then $D_1^\sigma \cap \dots \cap D_n^\sigma = \{A\}$, otherwise the intersection $D_1 \cap \dots \cap D_n$ would contain a component of positive dimension, which contradicts $D_1 \cap \dots \cap D_n \subseteq \text{St}_0(X)$.

Now we can define the combinatorial coefficient of the covering D_1, \dots, D_n at each point $x \in \text{St}_0(X)$ using Definition 1.3.1 of Chapter 1.

Definition 2.3.3. Let (X, D) be toroidal and (D_1, \dots, D_n) a reasonable covering of D . Then for a point $x \in \text{St}_0(X)$ the *combinatorial coefficient of the covering at x* is the combinatorial coefficient of the covering of σ in an equipped local model at x , corresponding to the covering (D_1, \dots, D_n) . We denote it by $c(x)$.

Remark 2.3.2. Invariance. It follows from Remark 1.3.1 that the combinatorial coefficient is the same for any two equipped local models that correspond to an automorphism of \mathbb{T} that preserves the form ω_0 , and changes sign otherwise. Also it is skew-symmetric in D_1, \dots, D_n .

In the next definition we introduce the toric symbol associated with a reasonable covering of D . This toric symbol becomes independent of the choice of ω_0 in an equipped local model.

Definition 2.3.4. Let (X, D) be toroidal, (D_1, \dots, D_n) a reasonable covering of D , and $x \in \text{St}_0(X)$. For $n + 1$ rational functions f_1, \dots, f_{n+1} on X with the support of their divisors in D define the *toric symbol at x associated with the covering* to be the $c(x)$ -th power of the toric symbol: $[f_1, \dots, f_{n+1}]_x^{c(x)}$.

Remark 2.3.3. Invariance. It follows from Remark 2.3.1 and Remark 2.3.2 that the toric symbol associated with the covering is already independent of the choice of ω_0 in an equipped local model at x . Furthermore, it is multiplicatively skew-symmetric in both f_1, \dots, f_{n+1} and D_1, \dots, D_n .

Now we will give a relation between the toric symbol associated with a covering and Parshin's tame symbol. Consider the stratification defined by the irreducible components of D (see Section 2.3.1). Let Z be a stratum. We say that the closure \overline{Z} has *signature* $\{i_1, \dots, i_m\}$ if $Z \subseteq D_{i_1} \cap \dots \cap D_{i_m}$ and m is maximal.

Proposition 2.3.2. Let (X, D) be toroidal and (D_1, \dots, D_n) a reasonable covering of D . For $x \in \text{St}_0(X)$ let $\mathcal{F}(x)$ be the set of all complete flags

$$x = X_0 \subset X_1 \subset \dots \subset X_{n-1} \subset X,$$

where $X_i \in \overline{\text{St}}_i(X)$ is a stratum closure of signature $\{i+1, \dots, n\}$, $0 \leq i \leq n-1$.

Then for any $n+1$ rational functions f_1, \dots, f_{n+1} with the support of their divisors in D we have

$$[f_1, \dots, f_{n+1}]_x^{c(x)} = \prod_{F \in \mathcal{F}(x)} \langle f_1, \dots, f_{n+1} \rangle_F,$$

where we assume that if $\mathcal{F}(x)$ is empty then the product is 1.

Proof. Since the definition of the toric symbol and Parshin's tame symbol are local we can pass to a local model at x . The statement then follows from Proposition 2.2.1 and Theorem 1.3.1. \square

2.3.4 Main theorem

Theorem 2.3.3. *Let X be a complete normal n -dimensional variety over an algebraically closed field k , and D a closed subset of X such that the pair (X, D) is toroidal.*

Let (D_1, \dots, D_n) be a reasonable covering of D such that each D_i is a disjoint union of two closed subsets of pure codimension 1:

$$D = D_1 \cup \dots \cup D_n, \quad D_1 \cap \dots \cap D_n \subseteq \text{St}_0(X), \quad D_i = D'_i \sqcup D''_i, \quad 1 \leq i \leq n. \quad (2.3.2)$$

We get 2^n disjoint finite closed subsets of X :

$$S_k = G_1 \cap \dots \cap G_n, \quad \text{where } G_i = D'_i \text{ or } D''_i, \quad 1 \leq i \leq n, \quad 1 \leq k \leq 2^n.$$

Then for any $n+1$ rational functions f_1, \dots, f_{n+1} on X with supports of their divisors in D the following 2^n numbers are equal:

$$\left(\prod_{x \in S_1} [f_1, \dots, f_{n+1}]_x^{c(x)} \right)^{(-1)^{|S_1|}} = \dots = \left(\prod_{x \in S_{2^n}} [f_1, \dots, f_{n+1}]_x^{c(x)} \right)^{(-1)^{|S_{2^n}|}}, \quad (2.3.3)$$

where $[f_1, \dots, f_{n+1}]_x$ denotes the toric symbol of f_1, \dots, f_{n+1} at x , $c(x)$ is the combinatorial coefficient at x , and $|S_k|$ is the number of D''_i in the definition of S_k .

Proof. Because of the symmetry it is sufficient to prove the equality for any two sets

$$S_1 = D'_1 \cap \cdots \cap D'_i \cap \cdots \cap D'_n \quad \text{and} \quad S_2 = D'_1 \cap \cdots \cap D''_i \cap \cdots \cap D'_n.$$

Since the toric symbol associated with the covering is multiplicatively skew-symmetric in D_1, \dots, D_n we may assume that $i = 1$, so

$$S_1 = D'_1 \cap D'_2 \cdots \cap D'_n \quad \text{and} \quad S_2 = D''_1 \cap D'_2 \cdots \cap D'_n.$$

We have to show that

$$\prod_{x \in S_1 \cup S_2} [f_1, \dots, f_{n+1}]_x^{c(x)} = 1.$$

Let Σ be the union of all components $Y \in \overline{\text{St}}_1(X)$ with signature $\{2', \dots, n'\}$. It follows from Proposition 2.3.2 that if $x \in S_1 \cup S_2$ does not lie on any component of Σ then the toric symbol $[f_1, \dots, f_{n+1}]_x^{c(x)}$ is 1. On the other hand, by (2.3.2) the signature of every point $x \in \text{St}_0(X) \cap \Sigma$ is either

$$\{2', \dots, n'\}, \quad \text{or} \quad \{1', 2', \dots, n'\}, \quad \text{or} \quad \{1'', 2', \dots, n'\}.$$

In the first case $[f_1, \dots, f_{n+1}]_x^{c(x)} = 1$, again by Proposition 2.3.2. In the second case $x \in S_1$ and in the third $x \in S_2$. Therefore, we have

$$\prod_{x \in S_1 \cup S_2} [f_1, \dots, f_{n+1}]_x^{c(x)} = \prod_{x \in \text{St}_0(X) \cap \Sigma} [f_1, \dots, f_{n+1}]_x^{c(x)}. \quad (2.3.4)$$

Now consider a component Y of Σ , and a point $y \in Y$. Let $\mathcal{F}(y, Y)$ denote the set of all complete flags

$$y \subset Y \subset X_2 \subset \cdots \subset X_{n-1} \subset X,$$

where X_i , $2 \leq i \leq n-1$ is the closure of an i -dimensional stratum with signature $\{(i+1)', \dots, n'\}$. Denote

$$\langle f_1, \dots, f_{n+1} \rangle_{y, Y} = \prod_{F \in \mathcal{F}(y, Y)} \langle f_1, \dots, f_{n+1} \rangle_F,$$

and we assume that $\langle f_1, \dots, f_{n+1} \rangle_{y,Y} = 1$ if $\mathcal{F}(y, Y)$ is empty. Then by Proposition 2.3.2 for each $x \in \text{St}_0(X) \cap \Sigma$ we have

$$[f_1, \dots, f_{n+1}]_x^{c(x)} = \prod_{F \in \mathcal{F}(x)} \langle f_1, \dots, f_{n+1} \rangle_F = \prod_{Y \ni x} \langle f_1, \dots, f_{n+1} \rangle_{x,Y}, \quad (2.3.5)$$

where the product on the right hand side runs over all components Y of Σ containing x .

On the other hand, by the first Parshin's reciprocity law (see Appendix A, Theorem A.3.2)

$$\prod_{y \in Y} \langle f_1, \dots, f_{n+1} \rangle_{y, Y \subset X_1 \subset X_2 \subset \dots \subset X_{n-1} \subset X} = 1,$$

where the product is taken over all points $y \in Y$. Thus

$$\prod_{y \in Y} \langle f_1, \dots, f_{n+1} \rangle_{y,Y} = 1.$$

Note that $\langle f_1, \dots, f_{n+1} \rangle_{y,Y}$ is trivial for all points y not lying in $\text{St}_0(X)$, so we can assume that $y \in \text{St}_0(X) \cap Y$. We have

$$\prod_{y \in \text{St}_0(X) \cap Y} \langle f_1, \dots, f_{n+1} \rangle_{y,Y} = 1. \quad (2.3.6)$$

Combining (2.3.4), (2.3.5) and (2.3.6) we get

$$\begin{aligned} \prod_{x \in S_1 \cup S_2} [f_1, \dots, f_{n+1}]_x^{c(x)} &= \prod_{x \in \text{St}_0(X) \cap \Sigma} [f_1, \dots, f_{n+1}]_x^{c(x)} = \\ \prod_{x \in \text{St}_0(X) \cap \Sigma} \prod_{Y \ni x} \langle f_1, \dots, f_{n+1} \rangle_{x,Y} &= \prod_{Y \subset \Sigma} \prod_{x \in \text{St}_0(X) \cap Y} \langle f_1, \dots, f_{n+1} \rangle_{x,Y} = 1. \end{aligned}$$

□

Chapter 3

Toric residue

Let (X, D) be a toroidal pair. We consider rational n -forms ω on X which are regular in $X \setminus D$. As in the case of the toric symbol we use a local model to define the toric residue $\text{res}_x^{\mathbb{T}} \omega$ at each 0-dimensional intersection x of components of D .

Let $D = D_1 \cup \cdots \cup D_n$ be a covering of D by n sets, each D_i being a union of components of D such that $D_1 \cap \cdots \cap D_n$ is 0-dimensional. We show that the number $c(x) \text{res}_x^{\mathbb{T}} \omega$ (where $c(x)$ is the combinatorial coefficient at x) is independent of the choice of a local model at x . Then we prove the reciprocity for the toric residue. The proof is based on the description of $c(x)$ given in Chapter 1 and a relation between the toric residue and Parshin's residue (Proposition 3.2.2). The latter allows us to reduce the proof to the 1-dimensional residue formula.

3.1 Toric residue on toroidal varieties

As before we consider an n -dimensional normal variety X over an algebraically closed field k and a closed subset D of X such that (X, D) form a toroidal pair.

First we will define the toric residue for a local model at a point $x \in X$. Let $(X_\sigma, x_0, \omega_0)$ be an n -dimensional affine toric variety corresponding to a rational convex cone σ with an apex, x_0 is the closed orbit, and $\omega_0 = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}$, where x_1, \dots, x_n is a coordinate system for the torus \mathbb{T} (see Chapter 2).

Let $A = \hat{\mathcal{O}}_{X_\sigma, x_0}$ be the completion of the local ring of x_0 on X_σ , $B = A_S$ the

localization of A by the multiplicative subgroup S of all monomials. We consider the B -algebra Ω_B^n of differential n -forms that are regular in \mathbb{T} and the A -algebra Ω_A^n of regular differential n -forms.

If x_1, \dots, x_n is a coordinate system for \mathbb{T} we can identify every element f of B with a formal power series

$$f(x) = x^b \sum_{a \in \sigma \cap \mathbb{Z}^n} \lambda_a x^a, \quad b \in \mathbb{Z}^n.$$

Let $\omega \in \Omega_B^n$. Then we can write

$$\omega = f \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n},$$

for some $f \in B$.

Definition 3.1.1. The *toric residue* of a differential n -form $\omega \in \Omega_B^n$ is the constant term λ_{-b} in the formal power series of f . We denote it by $\text{res}^{\mathbb{T}} \omega$.

We have the following properties of the toric residue.

Proposition 3.1.1. Consider a differential n -form $\omega \in \Omega_B^n$. Then

1. If ω is exact then $\text{res}^{\mathbb{T}} \omega = 0$;
2. If $\omega \in \Omega_A^n$ then $\text{res}^{\mathbb{T}} \omega = 0$;
3. For any $u_1, \dots, u_n \in B^\times$

$$\text{res}^{\mathbb{T}} \frac{du_1}{u_1^{m_1}} \wedge \cdots \wedge \frac{du_n}{u_n^{m_n}} = 0,$$

unless all $m_i = 1$, $1 \leq i \leq n$;

4. The toric residue is independent of the choice of a coordinate system x_1, \dots, x_n ;
5. The toric residue is invariant under monomial transformations $x \mapsto x^Q$,
 $Q \in GL(n, \mathbb{Z})$ up to a factor $\det Q$.

Proof. ¹ 1. Let $\omega = d\eta$. Without loss of generality we can assume that

$$\eta = g dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.$$

Then

$$\omega = (-1)^{i-1} \frac{\partial g}{\partial x_i} x_1 \cdots x_n \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n}.$$

Let $g = \sum_a \lambda_a x^a$. Then

$$(-1)^{i-1} \frac{\partial g}{\partial x_i} x_1 \cdots x_n = \sum_{a: a_i \neq 0} a_i \lambda_a \frac{x^a}{x_i} x_1 \cdots x_n, \quad a = (a_1, \dots, a_n).$$

Clearly the constant term of the last series is zero.

2. Let $u_i = x^{a_i}$, $a_i \in \sigma \cap \mathbb{Z}^n$ be n regular functions, such that du_1, \dots, du_n are linearly independent. Then for every $\omega \in \Omega_A^n$

$$\omega = f du_1 \wedge \cdots \wedge du_n = f dx^{a_1} \wedge \cdots \wedge dx^{a_n} = f x^{a_1 + \cdots + a_n} \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n},$$

where $f \in A$. Clearly, the constant term of $f x^{a_1 + \cdots + a_n}$ is zero.

3. First assume that $\text{char}(k) = 0$. Suppose $m_i \neq 1$. Then

$$\frac{du_1}{u_1^{m_1}} \wedge \cdots \wedge \frac{du_n}{u_n^{m_n}} = d \left(\frac{(-1)^{i-1}}{1 - m_i} u_i^{1-m_i} \frac{du_1}{u_1^{m_1}} \wedge \cdots \wedge \frac{\widehat{du_i}}{u_i^{m_i}} \wedge \cdots \wedge \frac{du_n}{u_n^{m_n}} \right),$$

and the statement follows from part 1. In case of arbitrary characteristic note that the toric residue is a polynomial function in finitely many coefficients of the series $u_1, \dots, u_n \in B^\times$. This function is independent of the characteristic and vanishes when the characteristic is zero. Therefore it is identically zero².

4. Let u_1, \dots, u_n be another coordinate system for \mathbb{T} . Then $u_i = \phi_i x_i$, where $\phi_i \in A^\times$.

Consider $\omega \in \Omega_B^n$. Then

$$\omega = f(u) \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n},$$

¹The proof of parts 1, 3 and 4 is similar to the one given in [F-P] for Parshin's residue.

²Similar argument appears in [Se], p. 21.

and the residue with respect to (u_1, \dots, u_n) is

$$\text{res}_{(u_1, \dots, u_n)}^{\mathbb{T}} \omega = \lambda_0, \quad \text{where } f(u) = \sum_a \lambda_a u^a, \quad a \in \mathbb{Z}^n.$$

On the other hand by part 3 the residue of ω with respect to (x_1, \dots, x_n) is equal to

$$\text{res}_{(x_1, \dots, x_n)}^{\mathbb{T}} \omega = \text{res}_{(x_1, \dots, x_n)}^{\mathbb{T}} \lambda_0 \frac{du_1}{u_1} \wedge \dots \wedge \frac{du_n}{u_n} \quad (3.1.1)$$

Now for each $1 \leq i \leq n$

$$\frac{du_i}{u_i} = \frac{d\phi_i}{\phi_i} + \frac{dx_i}{x_i}.$$

Thus opening brackets in (3.1.1) we get

$$\text{res}_{(x_1, \dots, x_n)}^{\mathbb{T}} \omega = \text{res}_{(x_1, \dots, x_n)}^{\mathbb{T}} \lambda_0 \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n} + \sum_i \text{res}_{(x_1, \dots, x_n)}^{\mathbb{T}} \omega_i = \lambda_0,$$

where in ω_i at least one of $\frac{dx_j}{x_j}$ is replaced by $\frac{d\phi_j}{\phi_j}$. It is easy to check that the residue of every ω_i is zero. For instance, let $\phi_1 = \sum_a \mu_a x^a$, $a \in \sigma \cap \mathbb{Z}^n$. Then

$$\frac{d\phi_1}{\phi_1} \wedge \frac{dx_2}{x_2} \wedge \dots \wedge \frac{dx_n}{x_n} = \phi_1^{-1} \sum_a a_1 \mu_a x^a \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n}.$$

Clearly $\phi_1^{-1} \sum_a a_1 \mu_a x^a$ belongs to A and has zero constant term.

5. It follows from the fact that if $t_i = x^{q_i}$, $q_i \in \mathbb{Z}^n$ then

$$\frac{dt_1}{t_1} \wedge \dots \wedge \frac{dt_n}{t_n} = (\det Q) \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n},$$

and the observation that monomial transformations do not change the constant term of a series. \square

Now let (X, D) be toroidal, ω a rational n -form on X , regular in $X \setminus D$. For each point $x \in \text{St}_0(X)$ we define the toric residue of ω as follows.

Definition 3.1.2. The *toric residue* of a rational n -form ω regular in $X \setminus D$ at a point $x \in \text{St}_0(X)$ is the toric residue of its image in a local model at x . We denote it by $\text{res}_x^{\mathbb{T}} \omega$.

Remark 3.1.1. Invariance. As it follows from Proposition 2.3.1 and Proposition 3.1.1 the toric residue is the same for any two equipped local models that are related by an isomorphism preserving the form ω_0 and changes sign otherwise.

3.2 Parshin's residue and toric residue

In this section we will give a relation between Parshin's residue and the toric residue.

Let $(X_\sigma, x_0, \omega_0)$ be an equipped n -dimensional affine toric variety corresponding to a rational convex cone σ with an apex, x_0 is the closed orbit. We consider a complete flag of orbit closures in X_σ :

$$F : \quad x_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X_\sigma.$$

As before we assume that the cone σ is oriented in accordance with the choice of ω_0 . Thus the sign $\text{sgn } F$ of the flag F is defined (see the definition before Proposition 2.2.1). We have the following proposition.

Proposition 3.2.1. *Let $(X_\sigma, x_0, \omega_0)$ be an equipped affine toric variety with closed orbit x_0 . Then for any n -form ω regular in \mathbb{T} we have*

$$\text{res}_F \omega = \text{sgn } F \text{res}^{\mathbb{T}} \omega,$$

where $\text{res}_F \omega$ is Parshin's residue at a complete flag F of orbit closures, $\text{sgn } F$ is the sign of F , and $\text{res}^{\mathbb{T}} \omega$ is the toric residue.

Proof. We repeat the arguments of the proof of Proposition 2.2.1. Consider a complete flag of orbit closures in X_σ

$$F : \quad x_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X_\sigma$$

and let F_σ be the corresponding flag of faces of σ :

$$F_\sigma : 0 = \sigma_0 \subset \sigma_1 \subset \cdots \subset \sigma_{n-1} \subset \sigma_n = \sigma.$$

Let x_1, \dots, x_n be coordinates in \mathbb{T} , $M \cong \mathbb{Z}^n$. Inside each σ_i , $1 \leq i \leq n$ choose a lattice point $q_i \in \mathbb{Z}^n$ at the unit integer distance from σ_{i-1} . Then $t_i = x^{q_i}$ is a monomial change of coordinates in \mathbb{T} . By Proposition 3.1.1

$$\det Q \operatorname{res}^{\mathbb{T}} \omega = \operatorname{res}^{\mathbb{T}} f(t) \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}.$$

On the other hand the rational functions $t_i = x^{q_i}$, $1 \leq i \leq n$ give a system of local parameters corresponding to F (see Appendix A). Therefore, by the definition of Parshin's residue (see Definition A.2.1)

$$\operatorname{res}_F \omega = \text{const term } f(t) = \det Q \operatorname{res}^{\mathbb{T}} \omega.$$

It remains to note that $\det Q = \operatorname{sgn} F$. □

Now let (X, D) be a toroidal pair. We recall the definition of a reasonable covering of D and the combinatorial coefficient associated with it.

Definition 3.2.1. Let (X, D) be a toroidal pair. We say that (D_1, \dots, D_n) is a *reasonable covering of D* if

$$D = D_1 \cup \cdots \cup D_n,$$

where each D_i is a union of some irreducible components of D , and

$$D_1 \cap \cdots \cap D_n \subseteq \operatorname{St}_0(X).$$

Let (D_1, \dots, D_n) be a reasonable covering of D . Then at each $x \in \operatorname{St}_0(X)$ we get a covering of the boundary of the cone σ in an equipped local model $(X_\sigma, x_0, \omega_0)$ at x (see Section 2.3.3).

Definition 3.2.2. Let (X, D) be toroidal and (D_1, \dots, D_n) a reasonable covering of D . Then for a point $x \in \text{St}_0(X)$ the *combinatorial coefficient of the covering at x* is the combinatorial coefficient of the covering of σ in an equipped local model at x , corresponding to the covering (D_1, \dots, D_n) . We denote it by $c(x)$.

Remark 3.2.1. Invariance. It follows from Remark 1.3.1 that the combinatorial coefficient is the same for any two equipped local models that correspond to an automorphism of \mathbb{T} that preserves the form ω_0 , and changes sign otherwise. Also it is skew-symmetric in D_1, \dots, D_n .

In the next definition we introduce the toric residue associated with a reasonable covering of D . This toric residue becomes independent of the choice of ω_0 in an equipped local model.

Definition 3.2.3. Let (X, D) be toroidal, (D_1, \dots, D_n) a reasonable covering of D , and $x \in \text{St}_0(X)$. For a rational n -form ω on X regular in $X \setminus D$ define the *toric residue at x associated with the covering* to be the $c(x)$ -th multiple of the toric residue: $c(x) \text{res}_x^{\mathbb{T}} \omega$.

Remark 3.2.2. Invariance. It follows from Remark 3.1.1 and Remark 3.2.1 that the toric residue associated with the covering is already independent of the choice of ω_0 in an equipped local model at x . Furthermore, it is skew-symmetric in D_1, \dots, D_n .

Now we will give a relation between the toric residue associated with a covering and Parshin's residue. Consider the stratification defined by the irreducible components of D (see Section 2.3.1). Let Z be a stratum. We say that the closure \overline{Z} has *signature* $\{i_1, \dots, i_m\}$ if $Z \subseteq D_{i_1} \cap \dots \cap D_{i_m}$ and m is maximal.

Proposition 3.2.2. *Let (X, D) be toroidal and (D_1, \dots, D_n) a reasonable covering of D . For $x \in \text{St}_0(X)$ let $\mathcal{F}(x)$ be the set of all complete flags*

$$x = X_0 \subset X_1 \subset \dots \subset X_{n-1} \subset X,$$

where $X_i \in \overline{\text{St}}_i(X)$ is a stratum closure of signature $\{i+1, \dots, n\}$, $0 \leq i \leq n-1$.

Then for any rational n -form regular in $X \setminus D$ we have

$$c(x) \text{res}_x^{\mathbb{T}} \omega = \sum_{F \in \mathcal{F}(x)} \text{res}_F \omega,$$

where we assume that if $\mathcal{F}(x)$ is empty then the sum is 0.

Proof. Since the definition of the toric residue and Parshin's residue are local we can pass to a local model at x . The statement then follows from Proposition 3.2.1 and Theorem 1.3.1. \square

3.3 Reciprocity for the toric residue on toroidal varieties

Theorem 3.3.1. *Let X be a complete normal n -dimensional variety over an algebraically closed field k , and D a closed subset of X such that the pair (X, D) is toroidal.*

Let (D_1, \dots, D_n) be a reasonable covering of D such that each D_i is a disjoint union of two closed subsets of pure codimension 1:

$$D = D_1 \cup \dots \cup D_n, \quad D_1 \cap \dots \cap D_n \subseteq \text{St}_0(X), \quad D_i = D'_i \sqcup D''_i, \quad 1 \leq i \leq n. \quad (3.3.1)$$

We get 2^n disjoint finite closed subsets of X :

$$S_k = G_1 \cap \dots \cap G_n, \quad \text{where } G_i = D'_i \text{ or } D''_i, \quad 1 \leq i \leq n, \quad 1 \leq k \leq 2^n.$$

Then for any rational n -form ω on X , regular in $X \setminus D$, the following 2^n numbers are equal:

$$(-1)^{|S_1|} \sum_{x \in S_1} c(x) \text{res}_x^{\mathbb{T}} \omega = \dots = (-1)^{|S_{2^n}|} \sum_{x \in S_{2^n}} c(x) \text{res}_x^{\mathbb{T}} \omega,$$

where $\text{res}_x^{\mathbb{T}} \omega$ denotes the toric residue of ω at x , $c(x)$ is the combinatorial coefficient at x , and $|S_k|$ is the number of D'_i in the definition of S_k .

Proof. The proof essentially repeats the arguments of the proof of Theorem 2.3.3, Chapter 2.

As before it is enough to prove the equality for the two sets

$$S_1 = D'_1 \cap D'_2 \cdots \cap D'_n \quad \text{and} \quad S_2 = D''_1 \cap D'_2 \cdots \cap D'_n.$$

We have to show that

$$\sum_{x \in S_1 \cup S_2} c(x) \text{res}_x^{\mathbb{T}} \omega = 0.$$

Let Σ be the union of all components $Y \in \overline{\text{St}}_1(X)$ with signature $\{2', \dots, n'\}$. It follows from Proposition 3.2.2 that if $x \in S_1 \cup S_2$ does not lie on any component of Σ then the toric residue $c(x) \text{res}_x^{\mathbb{T}} \omega$ is 0. On the other hand, by (3.3.1) the signature of every point $x \in \text{St}_0(X) \cap \Sigma$ is either

$$\{2', \dots, n'\}, \quad \text{or} \quad \{1', 2', \dots, n'\}, \quad \text{or} \quad \{1'', 2', \dots, n'\}.$$

In the first case $c(x) \text{res}_x^{\mathbb{T}} \omega = 0$, again by Proposition 3.2.2. In the second case $x \in S_1$ and in the third $x \in S_2$. Therefore, we have

$$\sum_{x \in S_1 \cup S_2} c(x) \text{res}_x^{\mathbb{T}} \omega = \sum_{x \in \text{St}_0(X) \cap \Sigma} c(x) \text{res}_x^{\mathbb{T}} \omega. \quad (3.3.2)$$

Now consider a component Y of Σ , and a point $y \in Y$. Let $\mathcal{F}(y, Y)$ denote the set of all complete flags

$$y \subset Y \subset X_2 \subset \cdots \subset X_{n-1} \subset X,$$

where X_i , $2 \leq i \leq n-1$ is the closure of an i -dimensional stratum with signature $\{(i+1)', \dots, n'\}$. Denote

$$\text{res}_{y,Y} \omega = \sum_{F \in \mathcal{F}(y,Y)} \text{res}_F \omega,$$

and we assume that $\text{res}_{y,Y} \omega = 0$ if $\mathcal{F}(y,Y)$ is empty. Then by Proposition 3.2.2 for each $x \in \text{St}_0(X) \cap \Sigma$ we have

$$c(x) \text{res}_x^{\mathbb{T}} \omega = \sum_{F \in \mathcal{F}(x)} \text{res}_F \omega = \sum_{Y \ni x} \text{res}_{x,Y} \omega, \quad (3.3.3)$$

where the sum on the right hand side runs over all components Y of Σ containing x .

On the other hand, by the first Parshin's reciprocity law (see Appendix A, Theorem A.3.4)

$$\sum_{y \in Y} \text{res}_{y \subset Y \subset X_2 \subset \dots \subset X_{n-1} \subset X} \omega = 0,$$

where the sum is taken over all points $y \in Y$. Thus

$$\sum_{y \in Y} \text{res}_{y,Y} \omega = 0.$$

Note that $\text{res}_{y,Y} \omega$ is trivial for all points y not lying in $\text{St}_0(X)$, so we can assume that $y \in \text{St}_0(X) \cap Y$. We have

$$\sum_{y \in \text{St}_0(X) \cap Y} \text{res}_{y,Y} \omega = 0. \quad (3.3.4)$$

Combining (3.3.2), (3.3.3) and (3.3.4) we get

$$\begin{aligned} \sum_{x \in S_1 \cup S_2} c(x) \text{res}_x^{\mathbb{T}} \omega &= \sum_{x \in \text{St}_0(X) \cap \Sigma} c(x) \text{res}_x^{\mathbb{T}} \omega = \\ \sum_{x \in \text{St}_0(X) \cap \Sigma} \sum_{Y \ni x} \text{res}_{x,Y} \omega &= \sum_{Y \subset \Sigma} \sum_{x \in \text{St}_0(X) \cap Y} \text{res}_{x,Y} \omega = 0. \end{aligned}$$

□

Chapter 4

Applications. Systems of equations

In this chapter we consider a system of n Laurent polynomials $P_1(x) = \cdots = P_n(x) = 0$, $x \in \mathbb{T} = (k^\times)^n$. We assume that the Newton polyhedra of the P_i have sufficiently general mutual locations. Recently A. Khovanskii [Kh1] found a formula for the product of the roots of such a system (for $k = \mathbb{C}$). Also O. Gelfond and A. Khovanskii [G-Kh] obtained a formula for the sum of the values of a Laurent polynomial over the roots of such a system (for $k = \mathbb{C}$).

We apply the main results of Chapter 2 and 3 (Theorem 2.3.3, Theorem 3.3.1) to give a uniform proof of the above mentioned results and extend them to the case of arbitrary algebraically closed field k .

4.1 Product of roots

Let $P(x)$ be a Laurent polynomial in n variables x_1, \dots, x_n with coefficients in an algebraically closed field k :

$$P(x) = \sum_V P(V)x^V, \quad P(V) \in k, \quad x^V = x_1^{v_1} \cdots x_n^{v_n}, \quad V = (v_1, \dots, v_n) \in \mathbb{Z}^n.$$

The *Newton polyhedron* $\Delta(P)$ of P is the convex hull in \mathbb{R}^n of points V for which $P(V) \neq 0$.

Consider a system of n algebraic equations in the algebraic torus $\mathbb{T} = (k^\times)^n$:

$$P_1(x) = \cdots = P_n(x) = 0, \quad x \in \mathbb{T}, \quad (4.1.1)$$

where the P_i are Laurent polynomials with Newton polyhedra Δ_i . We assume that none of the P_i is a monomial (otherwise the solution set is empty), hence, none of the Δ_i is a point.

Assume that the polyhedra $\Delta_1, \dots, \Delta_n$ are developed (see Chapter 1, Section 1.3.2, Definition 1.3.2). Then the solution set of the system (4.1.1) consists of isolated points only, which we call the roots of the system, each root with some multiplicity. We want to find the product of the roots of the system counting multiplicities. This product is a point in \mathbb{T} which we denote by $\rho(P_1, \dots, P_n)$. To locate it we find the value of every character $\chi : \mathbb{T} \rightarrow k^\times$ at $\rho(P_1, \dots, P_n)$. In coordinates (x_1, \dots, x_n) a character is represented by a monomial

$$\chi(x_1, \dots, x_n) = x_1^{m_1} \cdots x_n^{m_n} = x^m, \quad m = (m_1, \dots, m_n) \in \mathbb{Z}^n.$$

Definition 4.1.1.¹ The *symbol* of P_1, \dots, P_n and a character $\chi = x^m$ at a vertex $A \in \Delta$ is the symbol of $n + 1$ monomials $[P_1(A_1)x^{A_1}, \dots, P_n(A_n)x^{A_n}, x^m]$, where $A = A_1 + \cdots + A_n$ is the decomposition of A , and $P_i(A_i)$ is the coefficient of x^{A_i} in P_i . We denote it by $[P_1, \dots, P_n, \chi]_A$.

The goal of this section is to prove the following formula for the product of the roots of the system (4.1.1) in the case when $\Delta_1, \dots, \Delta_n$ are developed.

Theorem 4.1.1. *Suppose the Newton polyhedra $\Delta_1, \dots, \Delta_n$ of the system (4.1.1) are developed. Then the value of a character χ at the product $\rho(P_1, \dots, P_n)$ of the roots of the system is given by*

$$\chi(\rho(P_1, \dots, P_n)) = \prod_{A \in \Delta} [P_1, \dots, P_n, \chi]_A^{c(A)},$$

¹This definition is due to A. Khovanskii ([Kh1]). In fact, the number defined in [Kh1] coincides with ours after raising it to the power of $(-1)^n$.

where the product is taken over all vertices A of the polyhedron $\Delta = \Delta_1 + \cdots + \Delta_n$ and $c(A)$ is the combinatorial coefficient at A .

Remark 4.1.1. This theorem was proved by A. Khovanskii in the case of $k = \mathbb{C}$ in [Kh1]. Our proof uses the main result of Chapter 2 and works for an arbitrary algebraically closed field k .

Proof. First we will show that it is sufficient to prove the theorem for a generic system with given Newton polyhedra. Consider an N -dimensional space of all coefficients of the system, where N is the number of all lattice points inside and on the boundary of all the Newton polyhedra $\Delta_1, \dots, \Delta_n$. Let U be an open subset defined by the condition that the coefficients of the vertices of the polyhedra are non-zero. Then since the polyhedra are developed the number of the roots of the system counting multiplicities is the same for all points in U . The number $\chi(\rho(P_1, \dots, P_n))$ being a symmetric function of all the roots of the system is a rational function in U of the coefficients of the system. On the other hand, the product of the symbols $[P_1, \dots, P_n, \chi]_A$ is, clearly, a rational function of the coefficients of the system, and is regular in U . Assume now that we proved the formula for all systems in an open algebraic subset $W \subset U$. But two rational functions that coincide on an open algebraic subset $W \subset U$ coincide everywhere, thus we get the required result.

Let X be the complete toric variety associated with the polyhedron Δ (see for example [Da]). Let D be the closure of all codimension 1 orbits in X . Denote by Z_i the closure of the zero locus $P_i = 0$ in X , and let $Z = Z_1 \cup \cdots \cup Z_n$. By the above arguments we can assume that all the components of Z intersect transversally and the intersection of each component of Z with D is also transversal. In this case the pair $(X, D \cup Z)$ is toroidal.

Now we will define a covering of $D \cup Z$. Each irreducible component of D corresponds to a facet of Δ . Recall that each facet Γ of Δ has a unique decomposition into the sum of faces

$$\Gamma = \Gamma_1 + \cdots + \Gamma_n, \quad \text{where } \Gamma_i \subset \Delta_i, \quad i = 1, \dots, n. \quad (4.1.2)$$

Denote by D_i the union of all components that correspond to those facets whose i -th summand in the decomposition (4.1.2) is a vertex. Since the polyhedra $\Delta_1, \dots, \Delta_n$ are developed the sets D_1, \dots, D_n define a covering of D . Consider a covering

$$D \cup Z = (D_1 \cup Z_1) \cup \cdots \cup (D_n \cup Z_n).$$

By the transversality assumption and since the intersection $D_1 \cap \cdots \cap D_n$ consists of fixed orbits only, the covering is reasonable (see Chapter 2, Definition 2.3.2). Next we will show that it satisfies the condition of Theorem 2.3.3: $D_i \cap Z_i = \emptyset$.

Lemma 4.1.2. *For every $1 \leq i \leq n$, $D_i \cap Z_i = \emptyset$.*

Proof. Let E be a component of D_i . Then E is the union of some orbits of X . By the definition of D_i each such orbit \mathbb{O} corresponds to a face Γ of Δ whose i -th summand is a vertex A_i . Thus for each linear functional ξ such that ξ restricted to Δ achieves its maximum at Γ , the main term of $P_i(t^\xi)$ as $t \rightarrow \infty$ is the monomial $ct^{\xi(A_i)}$. Therefore $\mathbb{O} \cap Z_i = \emptyset$. \square

Now we apply Theorem 2.3.3 to get

$$\prod_{x \in Z_1 \cap \cdots \cap Z_n} [P_1, \dots, P_n, \chi]_x^{c(x)} = \prod_{x \in D_1 \cap \cdots \cap D_n} [P_1, \dots, P_n, \chi]_x^{(-1)^n c(x)}.$$

It remains to notice that for each transversal intersection x of components of Z the combinatorial coefficient $c(x) = 1$ and $[P_1, \dots, P_n, \chi]_x = \chi(x)^{(-1)^n \mu(x)}$ (see

Appendix A, Example A.1.1). Also for a point $x \in D_1 \cap \cdots \cap D_n$ the toric symbol $[P_1, \dots, P_n, \chi]_x$ coincides with $[P_1, \dots, P_n, \chi]_A$, where A is the corresponding vertex of Δ , and $c(x) = c(A)$, by definition. \square

4.2 Sum of residues

In this section we compute the sum of the values of a Laurent polynomial over the roots of a system of n algebraic equations in the algebraic torus $\mathbb{T} = (k^\times)^n$:

$$P_1(x) = \cdots = P_n(x) = 0, \quad x \in \mathbb{T}, \quad (4.2.1)$$

where the P_i are Laurent polynomials with Newton polyhedra Δ_i . As before we assume that the polyhedra Δ_i are developed.

The following definitions are due to A. Khovanskii and O. Gelfond [G-Kh]. Let P be a Laurent polynomial with Newton polygon $\Delta(P)$. For any Laurent polynomial Q define the Laurent series of the rational function Q/P at a vertex A of $\Delta(P)$ in the following way.

Let $P(A)$ be the coefficient in P of the monomial x^A corresponding to a vertex $A \in \Delta(P)$. Then the constant term of the Laurent polynomial $P' = P/(P(A)x^A)$ equals 1. Thus we can write

$$\frac{1}{P'} = 1 + (1 - P') + (1 - P')^2 + \dots \quad (4.2.2)$$

Note that each monomial appears only in a finite number of terms $(1 - P')^i$ and, thus, the coefficient of each monomial of this series is well-defined.

The formal product of the series (4.2.2) and the Laurent polynomial $Q/(P(A)x^A)$ is called the *Laurent series of Q/P at the vertex $A \in \Delta(P)$* .

Definition 4.2.1. The *residue* $\text{res}_A \omega$ at a vertex $A \in \Delta(P)$ of a rational n -form $\omega = \frac{Q}{P} \left(\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \right)$ is the constant term of the Laurent series of Q/P at A .

Theorem 4.2.1. *If $\Delta_1, \dots, \Delta_n$ are developed then the sum of the values of any Laurent polynomial R over the roots of the system counting multiplicities is given by*

$$\sum_x \mu(x)R(x) = (-1)^n \sum_{A \in \Delta} c(A) \operatorname{res}_A \left(\frac{x_1 \dots x_n R}{P_1 \dots P_n} J_P \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n} \right),$$

where the sum is taken over all vertices A of the polyhedron $\Delta = \Delta_1 + \dots + \Delta_n$, J_P is the Jacobian of the polynomial map $P = (P_1, \dots, P_n)$, and $c(A)$ is the combinatorial coefficient at A .

Remark 4.2.1. This theorem was proved by O. Gelfond and A. Khovanskii in the case of $k = \mathbb{C}$ in [G-Kh]. Their proof is topological, whereas our proof uses the main result of Chapter 2 and works for an arbitrary algebraically closed field k .

Proof. Let X be the complete toric variety associated with the polyhedron Δ (see for example [Da]). Let D be the closure of all codimension 1 orbits in X . Denote by Z_i the closure of the zero locus $P_i = 0$ in X , and let $Z = Z_1 \cup \dots \cup Z_n$. As in the proof of Theorem 4.1.1 it is enough to prove the theorem for a generic system with given Newton polyhedra, since the sum of the values of a Laurent polynomial over the roots of the system is a rational function of the coefficients of the system. Thus we can assume that all the components of Z intersect transversally and the intersection of each component of Z with D is also transversal. In this case the pair $(X, D \cup Z)$ is toroidal.

As before we define a covering of $D \cup Z$

$$D \cup Z = (D_1 \cup Z_1) \cup \dots \cup (D_n \cup Z_n),$$

where D_i is the union of all components of D that correspond to those facets Γ of Δ whose i -th summand in the decomposition $\Gamma = \Gamma_1 + \dots + \Gamma_n$, $\Gamma_j \subset \Delta_j$, is a

vertex. This is a reasonable covering and it satisfies the condition of Theorem 3.3.1: $D_i \cap Z_i = \emptyset$ (see Lemma 4.1.2).

Now we apply Theorem 3.3.1 to the form $\omega = R \frac{dP_1}{P_1} \wedge \cdots \wedge \frac{dP_n}{P_n}$ to get

$$\sum_{x \in Z_1 \cap \cdots \cap Z_n} c(x) \operatorname{res}_x^{\mathbb{T}} \omega = (-1)^n \sum_{x \in D_1 \cap \cdots \cap D_n} c(x) \operatorname{res}_x^{\mathbb{T}} \omega.$$

It remains to notice that for each transversal intersection x of components of Z the combinatorial coefficient $c(x) = 1$ and $\operatorname{res}_x^{\mathbb{T}} \omega = \mu(x)R(x)$ (see Appendix A, Example A.2.1). Also for a point $x \in D_1 \cap \cdots \cap D_n$, $c(x) = c(A)$ and

$$\operatorname{res}_x^{\mathbb{T}} \omega = \operatorname{res}_A \left(\frac{x_1 \cdots x_n R}{P_1 \cdots P_n} J_P \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_n}{x_n} \right),$$

where $A \in \Delta$ is the vertex corresponding to the fixed orbit x .

□

Appendix A

Parshin's Reciprocity Laws

Here we recall the definition of Parshin's tame symbol and residue for an arbitrary algebraic variety X over an algebraically closed field. We formulate Parshin's reciprocity laws (Theorem A.3.1, Theorem A.3.1) and give a proof of the first reciprocity (Theorem A.3.2, Theorem A.3.4) that we used in the proof of our main results (Theorem 2.3.3 of Chapter 2 and Theorem 3.3.1 of Chapter 3). We also remark that unlike the general case, the reciprocity laws for toroidal pairs follow immediately from the geometric properties of convex cones.

A.1 Parshin's tame symbol

Let X be a complete algebraic variety over an algebraically closed field k .

Consider a complete flag of irreducible subvarieties of X :

$$F : X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X. \tag{A.1.1}$$

Let \bar{Y} denote the normalization of Y . We have the following commutative diagram:

$$\begin{array}{ccccccc}
 X_n & \supset & X_{n-1} & \supset & X_{n-2} & \supset & \cdots \cdots \supset & X_0 & \quad (\text{A.1.2}) \\
 \uparrow & & \uparrow & & \uparrow & & & \uparrow & \\
 \bar{X}_n & \supset & X_{n-1}^{\nu_1} & & & & & & \\
 & & \uparrow & & & & & & \\
 & & \bar{X}_{n-1}^{\nu_1} & \supset & X_{n-2}^{\nu_2} & & & & \\
 & & & & \uparrow & & & & \\
 & & & & \bar{X}_{n-2}^{\nu_2} & \supset & \cdots & & \\
 & & & & & & & & \\
 & & & & & & \cdots \supset & X_0^{\nu_n} &
 \end{array}$$

where for each $i = 1 \dots n$, $X_{n-i}^{\nu_i}$ (ν_i depends on ν_1, \dots, ν_{i-1}) are the irreducible subvarieties of the normalization $\bar{X}_{n-i+1}^{\nu_{i-1}}$ which are mapped to X_{n-i} .

For each fixed collection $\nu = (\nu_1, \dots, \nu_n)$, define a system of local parameters $(u_1, \dots, u_n)_\nu$ as follows. On the normalization \bar{X}_n there exists an open subset where the codimension 1 subvariety $X_{n-1}^{\nu_1}$ has a local equation u_n . In general, for every $i = 0 \dots n-1$, let u_{n-i} be a local equation (in some open subset) of the codimension 1 subvariety $X_{n-i-1}^{\nu_{i+1}} \subset \bar{X}_{n-i}^{\nu_i}$.

Next, for every rational function f on X define its *order* $(a_1, \dots, a_n)_\nu$ corresponding to ν . The definition is by induction.

First let a_n be the order of f along $X_{n-1}^{\nu_1}$. We can write

$$f = f^{(n-1)} u_n^{a_n}, \quad a_n \in \mathbb{Z}.$$

Let $\tilde{f}^{(n-1)}$ be the restriction of $f^{(n-1)}$ on $X_{n-1}^{\nu_1}$, and a_{n-1} be the order of this restriction along $X_{n-2}^{\nu_2}$,

$$\tilde{f}^{(n-1)} = f^{(n-2)} u_{n-1}^{a_{n-1}}, \quad a_{n-1} \in \mathbb{Z},$$

and so on. Finally,

$$\tilde{f}^{(1)} = f^{(0)} u_1^{a_1}, \quad a_1 \in \mathbb{Z}, \quad (\text{A.1.3})$$

where a_1 is the order of $\tilde{f}^{(1)}$ at $X_0^{\nu_n}$.

Definition A.1.1. Let f_1, \dots, f_{n+1} be rational functions on X . Fix a complete flag F of irreducible subvarieties (A.1.1). For each collection ν let $(a_{i1}, \dots, a_{in})_\nu$ be the order of f_i corresponding to ν . Denote $A = (a_{ij}) \in M_{n+1 \times n}(\mathbb{Z})$. We have the following non-zero element of k :

$$\langle f_1, \dots, f_{n+1} \rangle_\nu = (-1)^B \left(\prod_{i=1}^{n+1} f_i^{(-1)^{i+1} A_i} \right)(x), \quad (\text{A.1.4})$$

where A_i is the determinant of the matrix obtained from A by eliminating its i -th row, and

$$B = \sum_k \sum_{i < j} a_{ik} a_{jk} A_{ij}^k,$$

where A_{ij}^k is the determinant of the matrix obtained from A by eliminating its i -th and j -th rows and its k -th column. Define *Parshin's tame symbol* of f_1, \dots, f_{n+1} at the flag F to be the product

$$\langle f_1, \dots, f_{n+1} \rangle_F = \prod_{\nu} \langle f_1, \dots, f_{n+1} \rangle_\nu.$$

Note that the order of the rational function inside the large brackets in (A.1.4) is $(0, \dots, 0)$, hence, its value at x makes sense and is not zero.

Remark A.1.1. Let us associate with every rational function f on X a monomial $cu_1^{a_1} \dots u_n^{a_n}$, where u_1, \dots, u_n are local parameters, $(a_1, \dots, a_n)_\nu$ is the order of f corresponding to ν , and $c = f^{(0)}(X_0^{\nu_n})$. Then the number $\langle f_1, \dots, f_{n+1} \rangle_\nu$ in the definition is the symbol of the corresponding $n + 1$ monomials (see Definition 2.1.1).

Parshin's tame symbol does not depend on the choice of local parameters $(u_1, \dots, u_n)_\nu$. It is multiplicative and skew-symmetric (compare to Proposition 2.1.1).

Example A.1.1. Let f_1, \dots, f_n be rational functions on an algebraic variety X , whose zero loci $\{f_i = 0\}$ intersect transversely at a non-singular point $x \in X$. Denote by Z_i the irreducible component of $\{f_i = 0\}$ that contains x .

Let h be a rational function on X invertible in an open neighborhood of x . Then

$$\langle f_1, \dots, f_n, h \rangle_F = h(x)^{(-1)^n \mu},$$

where

$$F: \quad x = X_0 \subset X_1 \subset \dots \subset X_{n-1} \subset X, \quad X_i = Z_{i+1} \cap \dots \cap Z_n,$$

and $\mu = \mu_1 \dots \mu_n$ is the product of the multiplicities μ_i of f_i along Z_i .

Indeed, for the system of local parameters at x we can choose the local equations of Z_i at x . For the flag F the last row of the matrix A is zero and the first n rows form a lower triangular matrix with the multiplicities μ_i on the diagonal. Therefore, $A_{n+1} = \mu_1 \dots \mu_n$ and $A_j = 0$, $1 \leq j \leq n$. It is also not hard to see that $B = 0$ (for example one can use the description of B given in the proof of Proposition 2.1.1).

A.2 Parshin's residue

As before let X be a complete algebraic variety over an algebraically closed field k , and F a complete flag of irreducible subvarieties of X :

$$F: \quad X_0 \subset X_1 \subset \dots \subset X_{n-1} \subset X_n = X. \quad (\text{A.2.1})$$

For a fixed collection ν let $(u_1, \dots, u_n)_\nu$ be a system of local parameters, as before.

Consider a rational differential n -form on X . Let $\bar{\omega}$ be the image of ω on the normalization \bar{X} of X . The rational function u_n is a local equation of $X_{n-1}^{\nu_1} \subset \bar{X}$ at a generic point of $X_{n-1}^{\nu_1}$. At a generic point of $X_{n-1}^{\nu_1}$ the differentials du_1, \dots, du_n are linearly independent, and we can write

$$\bar{\omega} = f du_1 \wedge \dots \wedge du_n, \quad \text{where} \quad f = \sum_{i_n > N_n} f_{i_n} u_n^{i_n}.$$

The restriction of the form $f_{-1} du_1 \wedge \dots \wedge du_{n-1}$ onto $X_{n-1}^{\nu_1}$ makes sense and gives us a rational $(n-1)$ -form ω_{n-1} on $X_{n-1}^{\nu_1}$. Continuing in this way we arrive at a sequence

of rational $(n - i)$ -forms ω_{n-i} on $X_{n-i}^{\nu_i}$, $i = 1, \dots, n$, the last one being a number $\omega_0 = f_{-1, \dots, -1}$ at the point $X_0^{\nu_n}$. Note that this number is the coefficient of the series

$$f = \sum_{i_n \geq N_n} \sum_{i_{n-1} \geq N_{n-1}(i_n)} \cdots \sum_{i_1 \geq N_1(i_2, \dots, i_n)} f_{i_1, \dots, i_n} u_1^{i_1} \cdots u_n^{i_n}, \quad f_{i_1, \dots, i_n} \in k,$$

where we identify f with an element of the field $k((u_1)) \cdots ((u_n))$. (Here $K((t))$ denotes the field of the Laurent power series in t with coefficients in a field K .)

Definition A.2.1. Let ω be a rational n -form on X . Fix a complete flag F of irreducible subvarieties (A.2.1). For each collection ν , let $\text{res}_\nu \omega$ denote the number $f_{-1, \dots, -1}$ constructed above. Then *Parshin's residue* $\text{res}_F \omega$ at the flag F is the sum

$$\text{res}_F \omega = \sum_{\nu} \text{res}_\nu \omega.$$

Parshin's residue does not depend on the choice of local parameters $(u_1, \dots, u_n)_\nu$ (see [F-P]). The proof of this statement is similar to the proof we gave for the invariance of the toric residue in Proposition 3.1.1, Chapter 3.

Example A.2.1. Let f_1, \dots, f_n be rational functions on an algebraic variety X , whose zero loci $\{f_i = 0\}$ intersect transversely at a non-singular point $x \in X$. Denote by Z_i the irreducible component of $\{f_i = 0\}$ that contains x .

Let h be a rational function on X regular in an open neighborhood of x . Then

$$\text{res}_F h \frac{df_1}{f_1} \cdots \frac{df_n}{f_n} = \mu h(x),$$

where

$$F : \quad x = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X, \quad X_i = Z_{i+1} \cap \cdots \cap Z_n,$$

and $\mu = \mu_1 \cdots \mu_n$ is the product of the multiplicities μ_i of f_i along Z_i .

Indeed, for the system of local parameters at x we can choose the local equations u_i of Z_i at x . Then

$$\text{res}_F h \frac{df_1}{f_1} \cdots \frac{df_n}{f_n} = \text{res}_F h \frac{du_1^{\mu_1}}{u_1^{\mu_1}} \cdots \frac{du_n^{\mu_n}}{u_n^{\mu_n}} = \text{res}_F h \mu_1 \cdots \mu_n \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_n}{u_n} = h(x) \mu.$$

A.3 Reciprocity Laws

Now we will formulate Parshin's reciprocity laws for the tame symbol and the residue. We will give a proof of one of the reciprocity laws, the one we used in the proof of the main theorem. We start with the reciprocity for the symbol.

Theorem A.3.1. *Let X be a complete n -dimensional algebraic variety over an algebraically closed field k . Fix a partial flag of irreducible subvarieties*

$$X_0 \subset \cdots \subset \widehat{X}_i \subset \cdots \subset X_n = X,$$

where X_i is omitted. Then for any $n + 1$ rational functions f_1, \dots, f_{n+1} on X

$$\prod_{X_i} \langle f_1, \dots, f_{n+1} \rangle_F = 1,$$

where the product is taken over all irreducible subvarieties X_i that complete the flag

$$F : X_0 \subset \cdots \subset X_{i-1} \subset X_i \subset X_{i+1} \subset \cdots \subset X_n = X,$$

and the product is finite.

This theorem was first proved by A. Parshin (see [F-P]). J.-P. Brylinski and D. A. McLaughlin provided a proof that uses Deligne cohomology [B-M1, B-M2, B-M3]. In the proof of our main results we use the reciprocity law in the case when $i = 0$. For the sake of completeness we give a sketch of the proof of this reciprocity law.

Theorem A.3.2. *Let X be a complete irreducible n -dimensional algebraic variety over an algebraically closed field k . Fix a partial flag of irreducible subvarieties $X_1 \subset \cdots \subset X_n = X$. Then for any $n + 1$ rational functions f_1, \dots, f_{n+1} on X*

$$\prod_{X_0 \in X_1} \langle f_1, \dots, f_{n+1} \rangle_{X_0 \subset X_1 \subset \cdots \subset X_n} = 1,$$

and the product is finite.

Proof. Following [F-P] we reduce the proof to Weil's reciprocity law ([Se]). We use induction. If $\dim X = 1$ the statement is Weil's reciprocity law for the curve $X_1 = X$.

Suppose $\dim X = n$. Consider the commutative diagram (A.1.2). Fix a collection $\nu = (\nu_1, \dots, \nu_n)$. Let $(u_1, \dots, u_n)_\nu$ be a system of parameters corresponding to ν and $a_i = (a_{i1}, \dots, a_{in})_\nu$ the order of f_i corresponding to ν . Put $A = (a_{ij}) \in M_{(n+1) \times n}(\mathbb{Z})$.

For each ν_1 let $f_i^{\nu_1}$ be the restriction of $f_i u_n^{-a_{in}}$ to $X_{n-1}^{\nu_1}$. Then

$$\langle f_1, \dots, f_{n+1} \rangle_\nu = (-1)^{B'} \prod_{k=1}^{n+1} \langle f_1^{\nu_1}, \dots, \widehat{f_k^{\nu_1}}, \dots, f_{n+1}^{\nu_1} \rangle_{\nu'}^{(-1)^{n+k+1} a_{kn}}, \quad (\text{A.3.1})$$

where $B' = \sum_{i < j} a_{in} a_{jn} A_{ij}^n$, as before A_{ij}^n is the determinant of the matrix obtained from A by eliminating its i -th and j -th rows and its n -th column, and $\nu' = (\nu_2, \dots, \nu_n)$.

To see this note that $\langle f_1, \dots, f_{n+1} \rangle_\nu$ is defined similarly to the determinant using ‘‘cofactor expansion’’ along the first column in the $(n+1) \times (n+1)$ matrix

$$\begin{pmatrix} f_1 & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & & \vdots \\ f_{n+1} & a_{n+11} & \dots & a_{n+1n} \end{pmatrix},$$

but instead of multiplying f_i by the corresponding cofactor we raise f_i to the power of the cofactor. The formula (A.3.1) is then the analog of the cofactor expansion along the last column.

It follows from (A.3.1) that

$$\langle f_1, \dots, f_{n+1} \rangle_F = (-1)^{\sum_\nu B'} \prod_{k=1}^{n+1} \prod_{\nu_1} \left(\langle f_1^{\nu_1}, \dots, \widehat{f_k^{\nu_1}}, \dots, f_{n+1}^{\nu_1} \rangle_{F'} \right)^{(-1)^{n+k+1} a_{kn}},$$

where $F : X_0 \subset \dots \subset X_n$ and $F' : X_0 \subset \dots \subset X_{n-1}$.

By the inductive assumption for every ν_1 and for every $1 \leq k \leq n+1$ we have

$$\prod_{X_0 \in X_1} \langle f_1^{\nu_1}, \dots, \widehat{f_k^{\nu_1}}, \dots, f_{n+1}^{\nu_1} \rangle_{F'} = 1.$$

Therefore, it only remains to take care of the sign. We need to show that

$$\sum_{X_0 \in X_1} \sum_{\nu} B' = 0.$$

Indeed, fix ν . In each term $a_{in}a_{jn}A_{ij}^n$ of B' every monomial is divisible by exactly one of the entries of the first column of the matrix A . But when we change the point X_0 only elements of the first column of A change. Moreover, for each i the sum of a_{in} 's over all points X_0 is zero, since the sum of the orders of $\tilde{f}_i^{(1)}$ (see (A.1.3)) over all points of $X_1^{\nu_{n-1}}$ is zero. \square

Remark A.3.1. Note that for a toroidal pair (X, D) all Parshin's reciprocity laws (Theorem A.3.1) for $i > 0$ follow from Proposition 2.2.1 of Section 2.2. Indeed, consider a complete flag

$$F : X_0 \subset \cdots \subset X_i \subset \cdots \subset X_n$$

of irreducible subvarieties of X . We can pass to a local model at X_0 and assume that X is an affine toric variety and $D = X \setminus \mathbb{T}$. Then for any $n + 1$ rational functions f_1, \dots, f_{n+1} with divisors in D the symbol $\langle f_1, \dots, f_{n+1} \rangle_F$ is trivial unless F is a flag of orbit closures on X . But if we fix all X_j , $j \neq i$, and vary X_i there are only two such flags F (since for any face τ of a polyhedral cone there are only two codimension 1 faces of τ that contain a fixed codimension 2 face of τ), and the signs of these two flags are opposite. Now we can apply Proposition 2.2.1.

The following theorem is the additive reciprocity for the residue.

Theorem A.3.3. *Let X be a complete n -dimensional algebraic variety over an algebraically closed field k . Fix a partial flag of irreducible subvarieties*

$$X_0 \subset \cdots \subset \widehat{X}_i \subset \cdots \subset X_n = X,$$

where X_i is omitted. Then for any rational n -form ω on X

$$\sum_{X_i} \text{res}_F \omega = 0,$$

where the sum is taken over all irreducible subvarieties X_i that complete the flag

$$F : X_0 \subset \cdots \subset X_{i-1} \subset X_i \subset X_{i+1} \subset \cdots \subset X_n = X,$$

and the sum is finite.

We will give a proof of this theorem in the case $i = 0$.

Theorem A.3.4. *Let X be a complete irreducible n -dimensional algebraic variety over an algebraically closed field k . Fix a partial flag of irreducible subvarieties $X_1 \subset \cdots \subset X_n = X$. Then for any rational n -form ω on X regular in $X \setminus D$*

$$\sum_{X_0 \in X_1} \text{res}_{X_0 \subset X_1 \subset \cdots \subset X_n} \omega = 0,$$

and the sum is finite.

Proof. The assertion of the theorem follows from the 1-dimensional residue formula.

Indeed, consider a complete flag F of irreducible subvarieties:

$$X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X,$$

and the corresponding commutative diagram (A.1.2). In the definition of the residue for each collection $\nu = (\nu_1, \dots, \nu_n)$ we constructed a sequence of rational $(n-i)$ -forms $\bar{\omega}_{n-i}$ on $\bar{X}_{n-i}^{\nu_i}$, $i = 1, \dots, n$. We have

$$\text{res}_F \omega = \sum_{\nu} \text{res}_{\nu} \omega = \sum_{\nu} \text{res}_{X_0^{\nu_n}} \bar{\omega}_1,$$

where $\text{res}_{X_0^{\nu_n}} \bar{\omega}_1$ is the (usual) residue of the rational 1-form $\bar{\omega}_1$ on $\bar{X}_1^{\nu_{n-1}}$ at $X_0^{\nu_n}$.

Therefore,

$$\sum_{X_0 \in X_1} \text{res}_F \omega = \sum_{\nu} \sum_{x \in \bar{X}_1^{\nu_{n-1}}} \text{res}_x \bar{\omega}_1 = 0$$

by the residue formula for $\bar{\omega}_1$ on $\bar{X}_1^{\nu_{n-1}}$.

□

Remark A.3.2. Similar arguments as in Remark A.3.1 show that for a toroidal pair (X, D) all Parshin's reciprocity laws (Theorem A.3.3) for $i > 0$ follow from Proposition 3.2.1 of Section 3.2.

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