Lattice-point generating functions for free sums of polytopes

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Introduction — Free Sums

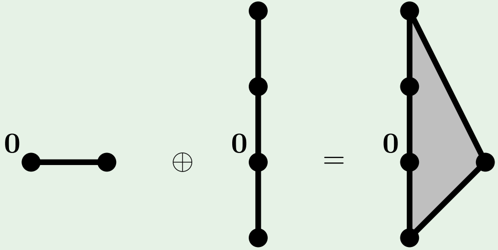
Notation: For $S \subseteq \mathbb{R}^n$, let $S_{\mathbb{Z}} = S \cap \mathbb{Z}^n$.

Definition

Let $\mathcal{P}, \mathcal{Q} \subseteq \mathbb{R}^n$ be polytopes. $\mathcal{P} \oplus \mathcal{Q} \coloneqq \operatorname{conv}(\mathcal{P} \cup \mathcal{Q})$ is a free sum if

- $\mathbf{0} \in \mathcal{P} \cap \mathcal{Q}$,
- $\operatorname{span} \mathcal{P} \oplus \operatorname{span} \mathcal{Q} = \mathbb{R}^n$ as a direct sum of vector spaces, and
- $(\operatorname{span} \mathcal{P})_{\mathbb{Z}} \oplus (\operatorname{span} \mathcal{Q})_{\mathbb{Z}} = \mathbb{Z}^n$ as a direct sum of lattices.

Example



Introduction — Free Sums

$\mathcal{P} \oplus \mathcal{Q}$ is related to familiar operations:

- Dual to cartesian product: Let $\mathcal{P}^\vee$ be the polar dual of $\mathcal{P}$ (with respect to span($\mathcal{P}$)). Then

$$(\mathcal{P} \oplus \mathcal{Q})^\vee = \mathcal{P}^\vee \times \mathcal{Q}^\vee.$$

- Homogenizes as Minkowski sum: Let cone($\mathcal{P}$) be the positive span of $\mathcal{P} \times \{1\} \subseteq \mathbb{R}^{n+1}$. Then

$$\mathrm{cone}(\mathcal{P} \oplus \mathcal{Q}) = \mathrm{cone}(\mathcal{P}) + \mathrm{cone}(\mathcal{Q}).$$

Introduction — Ehrhart Theory

Definition

$\mathcal{P} \subseteq \mathbb{R}^n$ is rational (resp., lattice) if all vertices are in $\mathbb{Q}^n$ (resp., $\mathbb{Z}^n$).
The denominator of $\mathcal{P}$ is $\operatorname{den}(\mathcal{P}) := \min\{k \in \mathbb{Z}_{\geq 1} : k\mathcal{P} \text{ is lattice}\}$.
The Ehrhart series of $\mathcal{P}$ is the formal power series

$$\operatorname{Ehr}_{\mathcal{P}}(t) := \sum_{k=0}^{\infty} |(k\mathcal{P})_{\mathbb{Z}}|\, t^k.$$

Theorem (E. Ehrhart 1962)

$\operatorname{Ehr}_{\mathcal{P}}(t)$ *is a rational function of the form*

$$\operatorname{Ehr}_{\mathcal{P}}(t) = \frac{\delta_{\mathcal{P}}(t)}{(1 - t^{\operatorname{den}(\mathcal{P})})^{\dim(\mathcal{P})+1}},$$

where $\delta_{\mathcal{P}}(t)$ is the δ-polynomial of $\mathcal{P}$.

Introduction — Braun's Theorem

Definition

Let V^* be the dual space of $V := \operatorname{span}(\mathcal{P})$. The polar dual of $\mathcal{P}$ is

$$\mathcal{P}^\vee := \{\varphi \in V^* : \varphi(\mathcal{P}) \subseteq \mathbb{R}_{\leq 1}\}.$$

Definition

Let $V_\mathbb{Z}^* := \{\varphi \in V^* : \varphi(V_\mathbb{Z}) \subseteq \mathbb{Z}\}$ be the dual lattice of $V_\mathbb{Z}$.

$\mathcal{P}^\vee$ is a lattice polyhedron in V^* if the vertices of $\mathcal{P}^\vee$ are in $V_\mathbb{Z}^*$.

Definition

A polytope $\mathcal{P}$ is reflexive if $\mathcal{P}$ and $\mathcal{P}^\vee$ are both lattice polytopes.

Braun's Theorem

Recall the δ-polynomial:

$$\mathrm{Ehr}_{\mathcal{P}}(t) = \sum_{k=0}^{\infty} |(k\mathcal{P})_{\mathbb{Z}}|\, t^k = \frac{\delta_{\mathcal{P}}(t)}{(1 - t^{\mathrm{den}(\mathcal{P})})^{\dim(\mathcal{P})+1}}.$$

Theorem (B. Braun 2006)

Let $\mathcal{P} \oplus \mathcal{Q}$ be a free sum. If $\mathcal{P}$ is a reflexive polytope and $\mathcal{Q}$ is a lattice polytope with $\mathbf{0} \in \mathcal{Q}^\circ$, then

$$\delta_{\mathcal{P}\oplus\mathcal{Q}}(t) = \delta_{\mathcal{P}}(t)\,\delta_{\mathcal{Q}}(t).$$

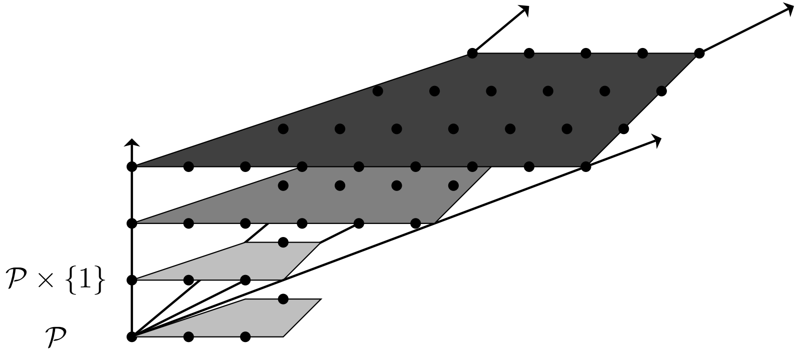
In terms of Ehrhart series, Braun's formula says

$$\mathrm{Ehr}_{\mathcal{P}\oplus\mathcal{Q}}(t) = (1-t)\,\mathrm{Ehr}_{\mathcal{P}}(t)\,\mathrm{Ehr}_{\mathcal{Q}}(t).$$

Introduction — cone($\mathcal{P}$)

Definition

Given polytope $\mathcal{P} \subseteq \mathbb{R}^n$, let cone($\mathcal{P}$) $\subseteq \mathbb{R}^{n+1}$ be the union of rays through $\mathcal{P} \times \{1\}$.



Introduction — Lattice-point Generating Functions

Definition

Let $\sigma_{\mathrm{cone}(\mathcal{P})}(\mathbf{x})$ be the lattice-point generating function of cone$(\mathcal{P})$:

$$\sigma_{\mathrm{cone}(\mathcal{P})}(\mathbf{x}) = \sigma_{\mathrm{cone}(\mathcal{P})}(x_1, \ldots, x_{n+1}) = \sum_{\mathbf{m} \in \mathrm{cone}(\mathcal{P})_{\mathbb{Z}}} \mathbf{x}^{\mathbf{m}},$$

where $\mathbf{x}^{\mathbf{m}} := x_1^{m_1} \cdots x_{n+1}^{m_{n+1}}$.

The Ehrhart series is a specialization of the lattice-point generating function:

$$\mathrm{Ehr}_{\mathcal{P}}(t) = \sigma_{\mathrm{cone}\,\mathcal{P}}(1, \ldots, 1, t).$$

First Main Result

Recall Braun's formula: If $\mathcal{P} \oplus \mathcal{Q}$ is a free sum with $\mathcal{P}$ reflexive and $\mathcal{Q}$ a lattice polytope such that $\mathbf{0} \in Q^\circ$, then

$$\operatorname{Ehr}_{\mathcal{P}\oplus\mathcal{Q}}(t) = (1-t)\operatorname{Ehr}_{\mathcal{P}}(t)\operatorname{Ehr}_{\mathcal{Q}}(t).$$

Theorem (Beck, M, Pallavi)

Let $\mathcal{P} \oplus \mathcal{Q} \subseteq \mathbb{R}^n$ be a free sum of rational polytopes. Then

$$\sigma_{\operatorname{cone}(\mathcal{P}\oplus\mathcal{Q})}(\mathbf{x}) = (1-x_{n+1})\,\sigma_{\operatorname{cone}\mathcal{P}}(\mathbf{x})\,\sigma_{\operatorname{cone}\mathcal{Q}}(\mathbf{x})$$

if and only if either $\mathcal{P}^\vee$ or $\mathcal{Q}^\vee$ is a lattice polyhedron.

- Implies Braun's theorem.
- Condition is necessary and sufficient.
- Doesn't require $\mathcal{P}$ or $\mathcal{Q}$ to be lattice. $\mathbf{0}$ can be on boundary of $\mathcal{P}$ or $\mathcal{Q}$.

Second Main Result

What if $\mathcal{P}$ and $\mathcal{Q}$ are rational, but neither $\mathcal{P}^\vee$ nor $\mathcal{Q}^\vee$ is a lattice polyhedron?

Definition

For integers $i \geq 0$, define the shifted cones

$$\operatorname{cone}^i \mathcal{P} := \operatorname{cone} \mathcal{P} + \frac{i}{\operatorname{den}(\mathcal{P}^\vee)} \mathbf{e}_{n+1},$$
$$\operatorname{cone}_i \mathcal{Q} := \operatorname{cone} \mathcal{Q} - \frac{i}{\operatorname{den}(\mathcal{P}^\vee)} \mathbf{e}_{n+1}.$$

Theorem (Beck, M, Pallavi)

Let $\mathcal{P} \oplus \mathcal{Q}$ be a free sum of rational polytopes. Then

$$\sigma_{\operatorname{cone}(\mathcal{P} \oplus \mathcal{Q})} = \sum_{i=0}^{\operatorname{den}(P^\vee)-1} \left(\sigma_{\operatorname{cone}^i \mathcal{P}} - \sigma_{\operatorname{cone}^{i+1} \mathcal{P}}\right) \sigma_{\operatorname{cone}_i \mathcal{Q}}.$$

Affine Free Sums

What if $\mathcal{P} \cap \mathcal{Q}$ isn't the origin? Suppose now that $\mathcal{P}, \mathcal{Q} \subseteq \mathbb{R}^n$ are polytopes such that

$$\mathcal{P} \cap \mathcal{Q} = \operatorname{aff}(\mathcal{P}) \cap \operatorname{aff}(\mathcal{Q}) = \{\mathbf{p}\}, \quad \mathbf{p} \in \mathbb{Q}^n.$$

Previously, $\mathcal{P} \oplus \mathcal{Q}$ was a free sum if

$$\mathbb{Z}^n = (\operatorname{span} \mathcal{P})_{\mathbb{Z}} \oplus (\operatorname{span} \mathcal{Q})_{\mathbb{Z}} \quad \text{(lattice direct sum)}.$$

Definition

Let $\Lambda^{\mathbf{p}} \subseteq \mathbb{R}^n$ be the lattice generated by $\mathbb{Z}^n \cup \{\mathbf{p}\}$.
For $S \subseteq \mathbb{R}^n$, let $S_{\Lambda^{\mathbf{p}}} := S \cap \Lambda^{\mathbf{p}}$.

$\mathcal{P} \oplus \mathcal{Q} := \operatorname{conv}(\mathcal{P} \cup \mathcal{Q})$ is an affine free sum of $\mathcal{P}$ and $\mathcal{Q}$ if

$$\Lambda^{\mathbf{p}} = \operatorname{span}(\mathcal{P} - \mathbf{p})_{\Lambda^{\mathbf{p}}} \oplus \operatorname{span}(\mathcal{Q} - \mathbf{p})_{\Lambda^{\mathbf{p}}}.$$

Third Main Result — Gorenstein polytopes

Definition

A lattice polytope $\mathcal{P}$ is Gorenstein of index k if there exists a (unique) point $\mathbf{m} \in (k\mathcal{P})_{\mathbb{Z}}$ such that $k\mathcal{P} - \mathbf{m}$ is reflexive.

Theorem (Beck, M., Pallavi)

Let polytope $\mathcal{P}$ be Gorenstein of index k with $k\mathcal{P} - \mathbf{m}$ reflexive. Let $\mathcal{Q} \subseteq \mathbb{R}^n$ be a polytope containing $\frac{1}{k}\mathbf{m} \in \mathcal{P}$ such that $\mathcal{P} \oplus \mathcal{Q}$ is an affine free sum. Then

$$\sigma_{\mathrm{cone}(\mathcal{P} \oplus \mathcal{Q})}(\mathbf{x}) = \left(1 - \mathbf{x}^{(\mathbf{m},k)}\right) \sigma_{\mathrm{cone}\,\mathcal{P}}(\mathbf{x})\, \sigma_{\mathrm{cone}\,\mathcal{Q}}(\mathbf{x}),$$

where $\mathbf{x}^{(\mathbf{m},k)} := x_1^{m_1} \cdots x_n^{m_n} x_{n+1}^{k}$.

Methods — Lattice-theory Lemmas

Recall: $\text{cone}(\mathcal{P} \oplus \mathcal{Q}) = \text{cone}(\mathcal{P}) + \text{cone}(\mathcal{Q})$

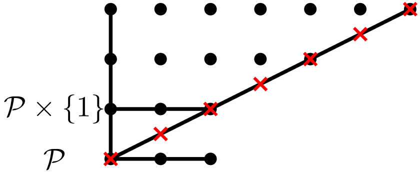
Equivalently,

$$\text{cone}(\mathcal{P} \oplus \mathcal{Q}) = \bigcup_{\mathbf{x} \in \text{cone}\,\mathcal{P}} (\mathbf{x} + \text{cone}\,\mathcal{Q})\,.$$

Definition

The lower envelope of $\underline{\partial}\,\text{cone}(\mathcal{P})$ is the subset of $\partial\,\text{cone}(\mathcal{P})$ that is vertically minimal.

The lower lattice envelope $\underline{\partial}_{\mathbb{Z}}\,\text{cone}(\mathcal{P})$ is the subset of points in $\underline{\partial}\,\text{cone}(\mathcal{P})$ that are directly beneath lattice points.



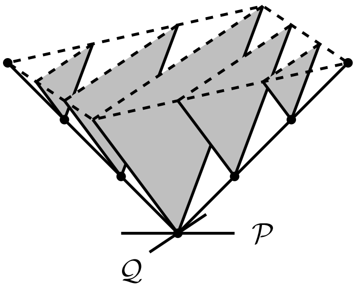
Methods — Lattice-theory Lemmas

Proposition

Let $\mathcal{P} \oplus \mathcal{Q}$ be a free sum of polytopes. Then

$$\operatorname{cone}(\mathcal{P} \oplus \mathcal{Q})_{\mathbb{Z}} = \bigsqcup_{\mathbf{m} \in \underline{\partial}_{\mathbb{Z}} \operatorname{cone} \mathcal{P}} (\mathbf{m} + \operatorname{cone} \mathcal{Q})_{\mathbb{Z}},$$

where $\bigsqcup$ denotes disjoint union.



Methods — Lattice-theory Lemmas

Proposition

Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a rational polytope containing the origin. Then the following are equivalent:

- $\mathcal{P}^\vee$ *is a lattice polyhedron,*
- $\underline{\partial}_{\mathbb{Z}} \operatorname{cone} \mathcal{P} = (\underline{\partial} \operatorname{cone} \mathcal{P})_{\mathbb{Z}}$,
- $(\underline{\partial} \operatorname{cone} \mathcal{P})_{\mathbb{Z}} = (\operatorname{cone} \mathcal{P})_{\mathbb{Z}} \setminus (\operatorname{cone} \mathcal{P} + \mathbf{e}_{n+1})_{\mathbb{Z}}$,
- $\sigma_{\underline{\partial} \operatorname{cone} \mathcal{P}}(\mathbf{x}) = (1 - x_{n+1})\, \sigma_{\operatorname{cone} \mathcal{P}}(\mathbf{x})$.

Open Questions

1. Is it possible to have $\underline{\partial}_{\mathbb{Z}} \operatorname{cone} \mathcal{P} = (\underline{\partial} \operatorname{cone} \mathcal{P})_{\mathbb{Z}}$ when $\mathcal{P}$ is a convex set other than a rational polytope?
2. "Dual to lattice" was the right generalization of "reflexive". What is the right generalization of "Gorenstein"? (Natural guess doesn't work.)