Short Tops and Semistable Fibrations

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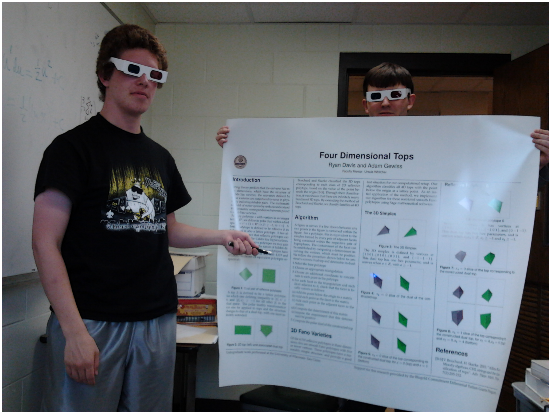


Polar Duality

Let $N \cong \mathbb{Z}^k$ be a lattice, with dual lattice $M = \mathrm{Hom}(N, \mathbb{Z})$. A lattice polytope has vertices in N.

Definition

Let Δ be a lattice polytope in N which contains $\vec{0}$. The polar polytope Δ° is the polytope in M given by:

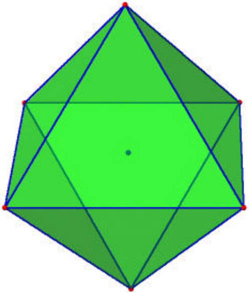
$$\{(m_1, \ldots, m_k) : (n_1, \ldots, n_k) \cdot (m_1, \ldots, m_k) \geq -1 \\ \text{for all } (n_1, \ldots, n_k) \in \Delta\}$$

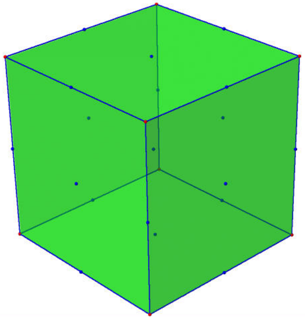
Reflexive Polytopes

Definition

A lattice polytope Δ is reflexive if Δ° is also a lattice polytope.

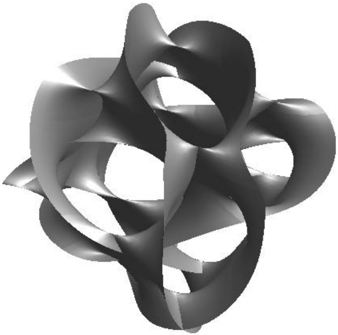
- If Δ is reflexive, $(\Delta^\circ)^\circ = \Delta$.
- Δ and Δ° are a mirror pair.





Calabi-Yau Varieties

- Anticanonical hypersurfaces inside toric varieties defined by fans over the faces of reflexive polytopes are Calabi-Yau varieties.
- For $k \leq 4$, we can resolve singularities torically, obtaining smooth elliptic curves, K3 surfaces, or Calabi-Yau threefolds.
- Mirror pairs of reflexive polytopes yield mirror families of Calabi-Yau varieties.



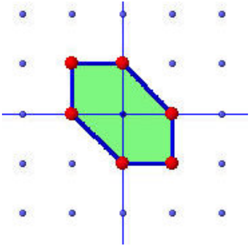
Classifying Reflexive Polytopes

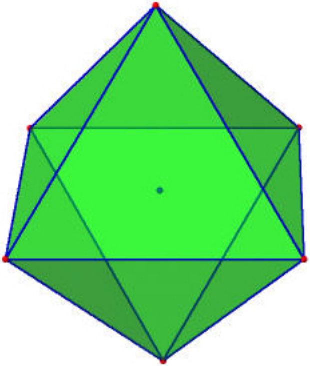
Up to a change of coordinates that preserves the lattice, there are . . .

Dimension	Reflexive Polytopes
1	1
2	16
3	4,319
4	473,800,776
5	??

Slicing Polytopes

Some reflexive polytopes contain lower-dimensional reflexive polytopes.





Tops

A top generalizes the idea of slicing a reflexive polytope:

Definition

A top is a lattice polytope Δ such that one of its defining inequalities is of the form

$$(n_1, \dots, n_k) \cdot (0, \dots, 0, 1) \geq 0$$

and the rest are of the form

$$(n_1, \dots, n_k) \cdot (m_{j1}, \dots, m_{jk}) \geq -1,$$

where $(m_{j1}, \dots, m_{jk})$ is a point in the lattice M.

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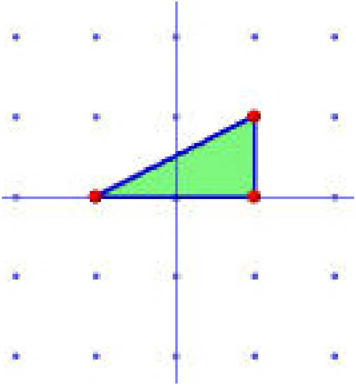
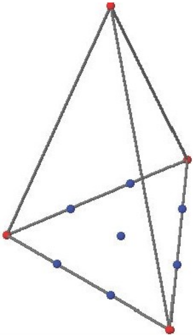
where $(m_{j1}, \ldots, m_{jk})$ is a point in the lattice M.

The reflexive boundary of a k-dimensional top is the $k-1$-dimensional reflexive polytope given by $\Delta \cap \{x_k = 0\}$.

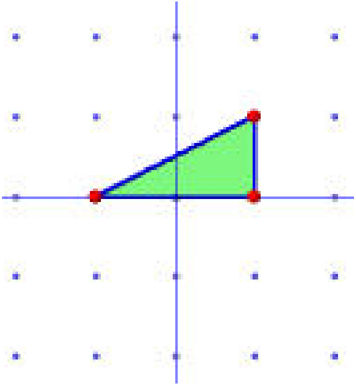
Examples of Tops

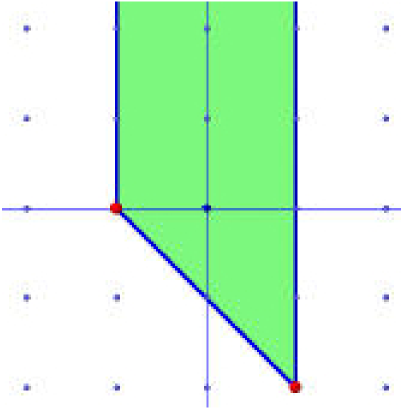




Dual Tops

The polar polytope of a top, also known as a dual top, is not bounded.



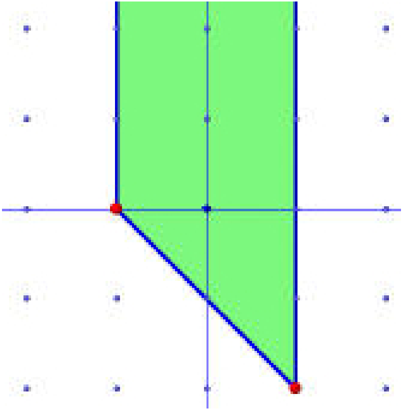


Dual Tops and Projection

A dual top also has a reflexive boundary facet. The vertical projection

$$(x_1, \ldots, x_k) \to (x_1, \ldots, x_{k-1})$$

has the reflexive boundary as image.



Classifying Tops

Vincent Bouchard and Harald Skarke classified three-dimensional tops.

- There are infinite families of isomorphism classes of tops.
- Not every top can be completed to a reflexive polytope.

The Bouchard and Skarke classification breaks into cases based on $z_{\min}$, the boundary point of the dual top which projects to the origin.
Classification of tops in dimensions 4 and higher is an open problem.

Tops and Fibrations

The fan over the faces of a top defines a fibration

$$F \hookrightarrow E \to \mathbb{C}.$$

The fiber F is the toric variety described by the reflexive boundary. We have an induced fibration of $k-1$-dimensional Calabi-Yau hypersurfaces

$$V \hookrightarrow W \to \mathbb{C}.$$

Degenerations

Definition

A degeneration is a morphism

$$\pi : X \to U$$

where:

- U is an open subset of $\mathbb{C}$ containing 0
- the restriction of π to $X \backslash \pi^{-1}(0)$ is smooth.

Tops and Degenerations

A top defines a degeneration of toric varieties when the toric variety defined by the reflexive boundary polytope is smooth. (We can always refine the fan appropriately for a reflexive boundary of dimension 3 or lower.)

A top defines a family of degenerations of Calabi-Yau hypersurfaces when the general anticanonical hypersurface in the toric variety defined by the reflexive boundary polytope is smooth. (We can always refine the fan appropriately for a reflexive boundary of dimension 4 or lower.)

Constructing Four-Dimensional Tops

Strategy

Describe dual tops with a fixed reflexive boundary.

- Vertices of the dual top must project to lattice points in the dual of the reflexive boundary.
- Use $GL(k, \mathbb{Z})$ to fix k vertices.
- Choose coordinates for the other vertices that will yield a convex polytope.

Local Convexity

Definition

We say a k-dimensional triangulated polytope $\diamond$ is locally convex if for every $k-2$-dimensional face f in the boundary of $\diamond$, the simplex defined by the two facets containing f is contained in $\diamond$.

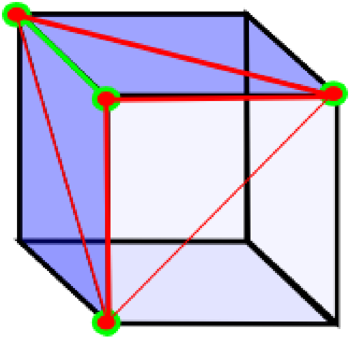


Figure: Image by Liam Flookes

A Determinantal Condition

We may check whether the simplices defined by $k-2$-dimensional faces f satisfy the local convexity condition by checking whether they are oriented consistently. If the points of the simplex are $\{(x_{11},\ldots,x_{1k}),\ldots,(x_{(k+1)1},\ldots,x_{(k+1)k})\}$, we use the determinant

$$\begin{vmatrix} x_{11} & \ldots & x_{1k} & 1 \\ \vdots & \ldots & \vdots & 1 \\ x_{(k+1)1} & \ldots & x_{(k+1)k} & 1 \end{vmatrix}$$

Short Tops

Definition

Let $\diamond$ be a top. The summit of $\diamond$ is the intersection of $\diamond$ with the half-space $x_k \geq 1$.

Definition

A short top is a top where the summit lattice points are contained in the hyperplane $x_k = 1$.

Duals of Short Tops

- $\diamond$ is a short top if and only if $\diamond^\circ$ contains the point $(0, \ldots, 0, -1)$.
- The summit of a short top $\diamond$ is a facet of the top if and only if the point $(0, \ldots, 0, -1)$ is a vertex of $\diamond^\circ$.

Constructing Short Tops

- Fix a dual boundary reflexive polytope Δ° and a triangulation of the boundary of Δ°.
- Assign a minimum x_k value of -1 to $(0, \ldots, 0) \in \Delta^\circ$.
- Assign a minimum x_k value of -1 to $k-1$ linearly independent vertices of Δ°.
- Assign minimum x_k values to the remaining lattice points of Δ°, satisfying local convexity. (Note that each non-vertical facet of the dual top must contain $(0, \ldots, 0, -1)$.)

Short Tops with Dual Simplex Base

Let Δ° be the $(k-1)$-dimensional simplex with vertices at $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$, and $(-1, \ldots, -1)$. Then Δ° defines a one-parameter family of short tops, with summit vertices given by

$$\begin{gathered}(a+1, 0, \ldots, 0, 1)\\ (0, a+1, \ldots, 0, 1)\\ \vdots\\ (0, 0, \ldots, a+1, 1)\end{gathered}$$

where $a \geq -1$ is an integer.

Short Tops with Dual Smooth Fano Base

We implemented a procedure in Sage to compute all isomorphism classes of smooth tops with dual reflexive base Δ° a 3-dimensional smooth Fano polytope. For a smooth Fano polytope with m vertices, we obtain an infinite family of tops with $m-3$ parameters.

Example: Reflexive Polytope and Polar Dual

Let Δ° be the smooth Fano polytope with vertices at $\begin{pmatrix} 1, & 0, & 0 \end{pmatrix}, \begin{pmatrix} 0, & 1, & 0 \end{pmatrix}, \begin{pmatrix} 0, & 0, & 1 \end{pmatrix}, \begin{pmatrix} -1, & 0, & 0 \end{pmatrix}$, and $\begin{pmatrix} 0, & -1, & -1 \end{pmatrix}$.

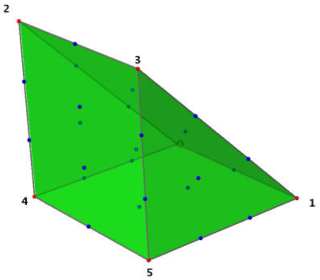


Figure: Δ

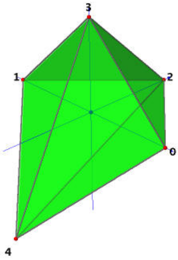


Figure: Δ°

Summit of the Top

The dual top $\diamond^\circ$ has two free parameters, and is convex when $a_1, a_2 \in \mathbb{Z}$, $a_1 \geq -1$ and $a_2 \geq -1$.

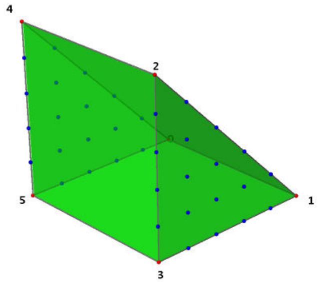


Figure: Summit of the short top for $a_1 = 0$ and $a_2 = 4$

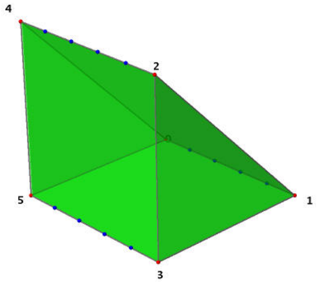


Figure: Summit of the short top for $a_1 = 4$ and $a_2 = 0$

Semistable Degenerations

Definition

A degeneration $\pi : X \to U \subset \mathbb{C}$ is semistable if X is nonsingular and $X_0 = \pi^{-1}(0)$ is reduced with normal crossings.

Short Tops and Semistable Degenerations

- A four-dimensional short top determines a family of semistable degenerations of K3 surfaces.
- A k-dimensional short top determines a family of semistable degenerations of $k-2$-dimensional Calabi-Yau varieties when we can resolve the fan over the faces of the top to a smooth fan.

Short Tops and Semistable Degenerations

- A four-dimensional short top determines a family of semistable degenerations of K3 surfaces.
- A k-dimensional short top determines a family of semistable degenerations of $k-2$-dimensional Calabi-Yau varieties when we can resolve the fan over the faces of the top to a smooth fan.

This construction differs from previous toric constructions of semistable degenerations (cf. [Hu06]), because we do not require our polytope to be simplicial, or its polar dual to be simple.

Semistable Degenerations of K3 Surfaces

Results of Kulikov, Persson, and Friedman and Morrison show that the minimal model of a semistable degeneration of K3 surfaces has the following structure. If $X_0 = \pi^{-1}(0)$, either:

- I X_0 is a smooth K3 surface.
- II X_0 is a chain of elliptic ruled components with rational surfaces at each end
- III X_0 consists of rational surfaces meeting along rational curves. The dual graph of X_0 has the sphere as topological support.

Short Tops and Type III Degenerations

Proposition

If $(0, 0, 0, -1)$ is a vertex of the polar dual of a 4-dimensional top, then the top is a short top which describes a Type III degeneration of K3 surfaces. The vertices of the dual graph of X_0 are given by the lattice points in the summit, and the topological support of the dual graph is the boundary of the summit, viewed as a 3-dimensional polytope.

Triangulating Reflexive Polytopes

Understanding regular triangulations of three-dimensional reflexive polytopes is the next step in a complete classification of four-dimensional tops.

For Further Reading

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