Triples of lattice polytopes with a given mixed volume

Ideals, Varieties, Applications

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Sparse Polynomial Systems and BKK theorem

CLO *Using Algebraic Geometry*, Section 7.5

Sparse Polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$:

$$f = \sum_{a \in \mathcal{A}} c_a x^a, \text{ where } x^a = x_1^{a_1} \cdots x_n^{a_n}, \quad c_a \in \mathbb{C}^*.$$

The set of exponents $\mathcal{A} \subset \mathbb{Z}^n$ is the support of f. The convex hull of the support $P = \text{conv}(\mathcal{A})$ is the Newton Polytope of f.

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Theorem (Kushnirenko 1975)

Let $f_1 = \cdots = f_n = 0$ be a generic sparse system with the same Newton polytope P. Then it has exactly $\text{Vol}_n(P)$ isolated solutions in $(\mathbb{C}^*)^n$.

Here $\text{Vol}_n(P)$ is the lattice volume of P, that is Euclidean n-dimensional volume normalized such that $\text{Vol}_n(\Delta) = 1$ for a unimodular simplex Δ.

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Theorem (Bernstein–Khovanskii–Kushnirenko 1976)

Let $f_1 = \cdots = f_n = 0$ be a generic sparse system with Newton polytopes $P_1, \ldots, P_n$. Then it has exactly $V(P_1, \ldots, P_n)$ isolated solutions in $(\mathbb{C}^*)^n$.

Here $V(P_1, \ldots, P_n)$ is the (lattice) mixed volume of the polytopes $P_1, \ldots, P_n$.

Mixed Volume: Definition and Properties

Recall the Minkowski sum $P + Q = \{p + q \in \mathbb{R}^n \mid p \in P, q \in Q\}$ for any $P, Q \subset \mathbb{R}^n$.

Mixed Volume is the coefficient of $t_1 \cdots t_n$ in the polynomial

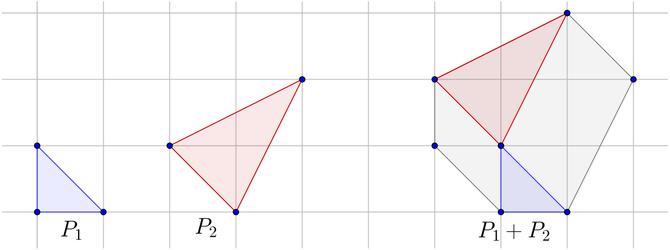
$$\mathrm{Vol}_n(t_1 P_1 + \cdots + t_n P_n) = \mathrm{Vol}_n(P_1) t_1^n + \cdots + V(P_1, \dots, P_n) t_1 \cdots t_n + \dots$$

Properties:

- symmetric, multilinear w.r.t. Minkowski addition
- $V(P, \dots, P) = \mathrm{Vol}_n(P)$
- $V(P_1, \dots, P_n) \geq 0$ (non-negativity)
- $V(P_1, \dots, P_n) \leq V(Q_1, \dots, Q_n)$ for $P_i \subseteq Q_i$ (monotonicity)

Mixed Volume: Example

Example: Consider P_1, P_2 in $\mathbb{R}^2$



We have
$V(P_1, P_2) = \frac{1}{2}\left(\mathrm{Vol}_2(P_1 + P_2) - \mathrm{Vol}_2(P_1) - \mathrm{Vol}_2(P_2)\right) = 4.$

Esterov's Question

Question: Given $m \in \mathbb{N}$ can one describe all n-tuples of lattice polytopes $(P_1, \ldots, P_n)$ such that a generic sparse system $f_1 = \cdots = f_n = 0$ with Newton polytopes $P_1, \ldots, P_n$ has exactly m solutions in $(\mathbb{C}^*)^n$?

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State of the art:

- (Esterov–Gusev '15) $m = 1$ and any $n \geq 1$
- (Esterov–Gusev '16) $m \leq 4$ and $n = 2$
- (Esterov–Gusev '16) $m \leq 4$, any $n \geq 1$, unmixed and spanning
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(Esterov, '19) The problem of describing all n-variate sparse systems that are solvable in radicals reduces to describing all k-variate sparse systems with up to 4 solutions, for $k \leq n$.

Combinatorial Problem

Problem: Given $m \in \mathbb{N}$ classify all n-tuples of lattice polytopes $(P_1, \ldots, P_n)$ with $V(P_1, \ldots, P_n) = m$.

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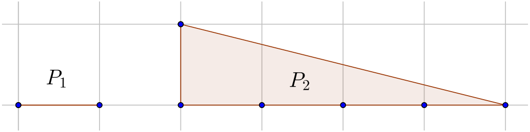
Warning: This is no longer true for n-tuples of polytopes!

Example:



$V(P_1, P_2) = 1$

Indeed, the system $f_1 = ax + c = 0$, $f_2 = y + h(x) = 0$ has 1 solution regardless of deg h.

Irreducible n-tuples

Reduction: If $P_1, \ldots, P_k \subset L$ for some k-subspace L

$$V(P_1, \ldots, P_k, \ldots, P_n) = V_L(P_1, \ldots, P_k) V_{\mathbb{R}^n/L}(\pi(P_{k+1}), \ldots, \pi(P_n)),$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^n/L$ is the projection along L.

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Definition: A tuple $(P_1, \dots, P_n)$ is irreducible if the sum of any k of the P_i has dimension greater than k, for $1 \le k < n$.

Theorem (Esterov-Gusev '18) There are finitely many irreducible n-tuples of lattice polytopes $(P_1, \dots, P_n)$ with a given mixed volume, up to lattice equivalence (i.e. $GL(n, \mathbb{Z})$ and independent translations).

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Idea: $\mathrm{Vol}_n(P_1 + \dots + P_n) < n^n m^{2^n}$, where $m = V(P_1, \dots, P_n)$.

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Idea: $\mathrm{Vol}_n(P_1 + \cdots + P_n) < n^n m^{2^n}$, where $m = V(P_1, \dots, P_n)$.

Challenge: For $n = 3$, $m = 4$ this bound is huge.
Moreover, the sharp upper bound must be at least $(n - 1 + m)^n$, since $V(\Delta, \dots, \Delta, m\Delta) = m$ for a unimodular simplex Δ.
For $n = 3, m = 4$ we get 216. There are $\sim 6,000,000$ polytopes of volume at most 36 (Balletti'18).

Our approach

- Enough to classify maximal triples (P_1, P_2, P_3)
- For this, employ relations between all possible mixed volumes $V(P_i, P_j, P_k)$, for $1 \leq i \leq j \leq k \leq 3$

Maximal n-tuples

Definition: A tuple $(P_1, \dots, P_n)$ is maximal in P_n if for any $P'_n \supsetneq P_n$ we have $V(P_1, \dots, P_n) < V(P_1, \dots, P'_n)$. A tuple $(P_1, \dots, P_n)$ is maximal if it is maximal in each P_i.

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$P_1, \ldots, P_{n-1}$ define mixed area measure $S_{P_1,\ldots,P_{n-1}}$ which is a finite measure on the set of primitive vectors u such that

$$S_{P_1,\ldots,P_{n-1}}(u) = V(P_1^u, \ldots, P_{n-1}^u), \text{ where } P_i^u = \text{face of } P_i \text{ in direction } u$$

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Proposition: If $(P_1, \dots, P_n)$ is maximal in P_n then

$$P_n = \operatorname{conv}\{x \in \mathbb{Z}^n : \langle x, u_i \rangle \le h_i, u_i \in \operatorname{supp} S_{P_1,\dots,P_{n-1}}\}$$

where the $h_i \in \mathbb{Z}_{\ge 0}$ satisfy

$$\sum h_i S_{P_1,\dots,P_{n-1}}(u_i) = V(P_1, \dots, P_n).$$

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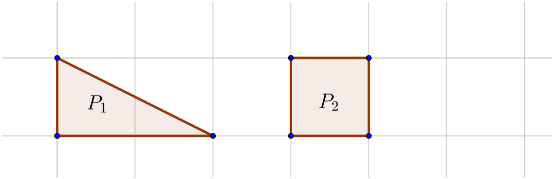

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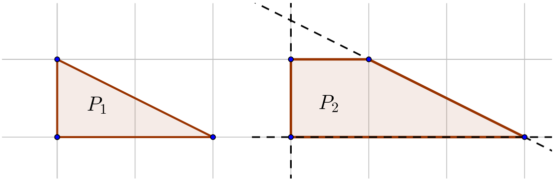

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Aleksandrov–Fenchel relations

Aleksandrov–Fenchel inequality:

$V(K, L, M)^2 \geq V(K, K, M)V(L, L, M)$ for convex bodies $K, L, M \subset \mathbb{R}^3$.

Denote $v_{ijk} = V(P_i, P_j, P_k)$ for $1 \leq i \leq j \leq k \leq 3$.

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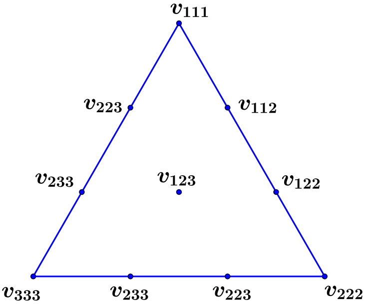

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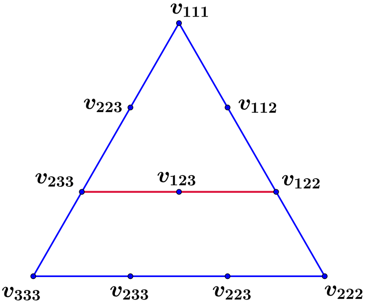

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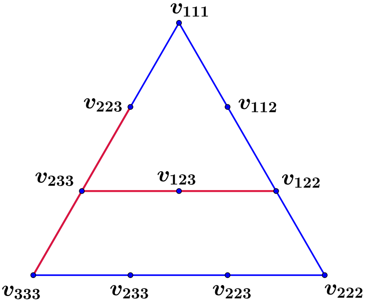

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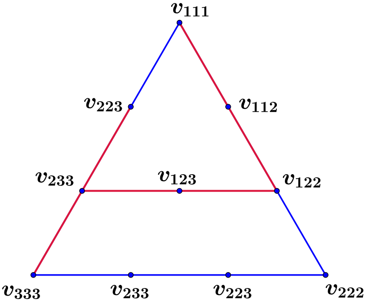

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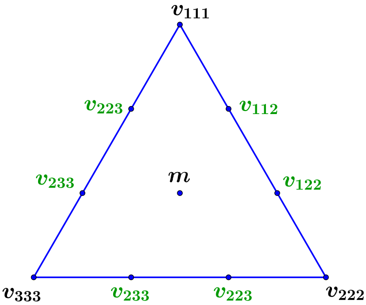
Special case: P_1, P_2, P_3 are 3-dimensional

The middle value $v_{123} = m$.



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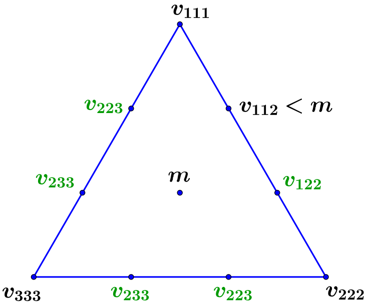
Either at least one of the green values is less than m ...



Recursively we can find (P_1, P_1, P_2).
Then find maximal P_3 as before.

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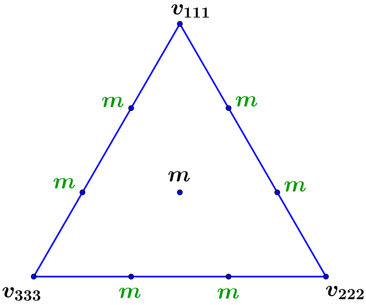
... or all of the green values are equal to m, by Aleksandrov–Fenchel.



Then we can show $P_1 = P_2 = P_3$.
Pick P_1 of volume m.

Output: number of triples with $V(P_1, P_2, P_3) = m$

m	# full-dim'l triples		# all maximal triples	running time
	unmixed			
1	1	1	1	
2	3	4	7	$\sim$ 2 hours
3	6	10	21	$\sim$ 1 day
4	17	30	92	$\sim$ 3 days

Pictures (and more) are here:
github.com/christopherborger/mixed_volume_classification

Further work

- Find a sharp upper bound on $\mathrm{Vol}_n(P_1 + \cdots + P_n)$ in terms of $m = V(P_1, \ldots, P_n)$.

  Conjecture: $\mathrm{Vol}_n(P_1 + \cdots + P_n) \leq (n - 1 + m)^n$ attained at $(\Delta, \ldots, \Delta, m\Delta)$.
  (True for $n = 2, 3$ in full-dim case. Also $\mathcal{O}(m^n)$ holds for $n \leq 6$.)

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- Is there a "structural" result across all n? For example,
  - (Hofscheier–Katthän–Nill '19) There are only finitely many spanning polytopes of given volume up to lattice equivalence and unit pyramid construction.
  - (Balletti–Borger'19) All n-tuples $(P_1, \ldots, P_n)$ with $V(P_1, \ldots, P_n) = (P_1 + \cdots + P_n)^\circ \cap \mathbb{Z}^n + 1$ are lattice projections onto $(\Delta_{n-1}, \ldots, \Delta_{n-1})$, except for finitely many exceptions.

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Thank you!