Toric codes and Minkowski length of polytopes

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What are toric codes? (reminder and setup)

- Let $\mathbb{T} = (\mathbb{F}_q^*)^r = \{p_1, \ldots, p_n\}$, the algebraic torus.
- Let P be a lattice polytope in $\mathbb{R}^r$.
  It defines a finite dimensional space of Laurent polynomials:

$$\mathcal{L}_P = \operatorname{span}_{\mathbb{F}_q}\{t^a \mid a \in P \cap \mathbb{Z}^r\}, \text{ where } t^a = t_1^{a_1} \cdots t_r^{a_r}.$$

Evaluation Map:

$$\operatorname{ev}_{\mathbb{T}} : \mathcal{L}_P \to \mathbb{F}_q^n \quad f \mapsto (f(p_1), \ldots, f(p_n)).$$

Toric Code: $\mathcal{C}_P = ev_{\mathbb{T}}(\mathcal{L}_P)$.

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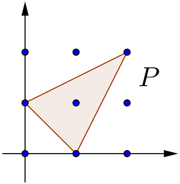

Let $\mathbb{F}_q = \mathbb{F}_4$ and $r = 2$. Then $|\mathbb{T}| = |(\mathbb{F}_q^*)^2| = 9$.

$$\mathcal{L}_P = \{\lambda_1 t_1 + \lambda_2 t_2 + \lambda_3 t_1 t_2 + \lambda_4 t_1^2 t_2^2 \mid \lambda_i \in \mathbb{F}_4\}.$$

In fact, $\mathcal{C}_P$ is a $[9, 4, 3]_4$-code.

Parameters and some properties of $\mathcal{C}_P$

D. Ruano (2007): The evaluation map $\mathrm{ev}_{\mathbb{T}} : \mathcal{L}_P \to \mathbb{F}_q^n$ is injective *iff* points in $P \cap \mathbb{Z}^r$ are distinct in $(\mathbb{Z}/(q-1)\mathbb{Z})^r$.

In this case $\mathcal{C}_P$ has parameters:
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J. Soprunova, — (2010) For any lattice polytopes P, Q

$$d(\mathcal{C}_{P\times Q}) = d(\mathcal{C}_P)d(\mathcal{C}_Q).$$

$$d(\mathcal{C}_{\ell Pyr(Q)}) = (q-1)d(\mathcal{C}_{\ell Q}),\ \text{for any } \ell = 1,2,3,\ldots$$

Largest number of zeros on hypersurfaces

Recall that

$$d(\mathcal{C}_P) = n - N_q(P),$$

where

$$N_q(P) = \max_{0 \neq f \in \mathcal{L}_P} |Z(f)| \quad \text{and} \quad Z(f) = \{p \in \mathbb{T} \mid f(p) = 0\},$$

the zero set of f in $\mathbb{T}$.

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Thus we should take f to be the product of linear factors, so

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attained at $f(t_1, \dots, t_r) = (t_1 - \alpha_1) \cdots (t_1 - \alpha_\ell)$ for distinct $\alpha_1, \dots, \alpha_\ell \in \mathbb{F}_q^*$.

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Similar principle holds in general for large q.

Connecting Lemma

Little-Schenck (2006) ($r = 2$); J. Whitney (2010) ($r = 3$)

Theorem

Let P be a lattice polytope in $\mathbb{R}^r$ and $q > \alpha(P)$ (large enough). For any $f, g \in \mathcal{L}_P$ consider factorizations into absolutely irreducible factors:

$$f = f_1 \cdots f_s \quad \text{and} \quad g = g_1 \cdots g_t,$$

and assume the g_i are distinct. Then $s < t$ implies $|Z(f)| < |Z(g)|$.

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Idea for $r = 2$: If Y is an irreducible projective curve over $\overline{\mathbb{F}}_q$ then

$$\text{Hasse-Weil Bound} \quad q + 1 - 2g\sqrt{q} \leq |Y(\mathbb{F}_q)| \leq q + 1 + 2g\sqrt{q}$$

where g is the genus of Y. On one hand,

$$|Z(f)| \leq \sum_{i=1}^{s} |Z(f_i)| \leq sq + \text{lower order terms}$$

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$$tq + \text{lower order terms} \leq \sum_{i=1}^{t} |Z(g_i)| - \sum_{i<j} |Z(g_i) \cap Z(g_j)| \leq |Z(g)|$$

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Main Questions:

1. What is the largest number of factors $f \in \mathcal{L}_P$ may have?
2. What do the irreducible factors in this case look like?
3. Can we bound the number of $\mathbb{F}_q$-zeros of irreducible factors?

Connecting Lemma

In fact, the first two questions are about the geometry of P.

1. The largest number of factors $f \in \mathcal{L}_P$ may have is the Minkowski length of P.
2. The irreducible factors of such f have Newton polytopes that are strongly indecomposable.

The third question is more algebraic.

3. Bound the maximum number of $\mathbb{F}_q$-zeros $N_q(P)$ where P is strongly indecomposable.

Once we know that we can bound the minimum distance of $\mathcal{C}_P$

Newton polytopes and Minkowski Sum

Let f be a Laurent polynomial $f \in \mathbb{F}_q[t_1, \ldots, t_r]$. Let $P(f)$ be its Newton Polytope: $P(f) =$ conv.hull $\{$ exponents of $f\} \subset \mathbb{R}^r$

Note: Newton polytope generalizes the notion of degree:

$$P(fg) = P(f) + P(g)$$

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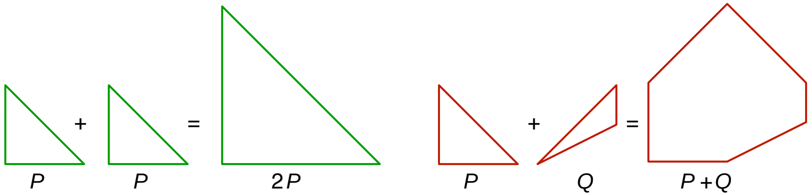
The Minkowski sum of polytopes P, Q in $\mathbb{R}^r$ is

$$P + Q = \{p + q \in \mathbb{R}^r \mid p \in P,\ q \in Q\}.$$



Minkowski length $L(P)$: Definition

Let P be a lattice polytope in $\mathbb{R}^r$.

Definition: The largest number of lattice polytopes of positive dimension whose Minkowski sum is contained in P is called the Minkowski length:

$$L(P) = \max\{L \in \mathbb{N} \mid Q = Q_1 + \cdots + Q_L \subseteq P, \dim Q_i > 0\}.$$

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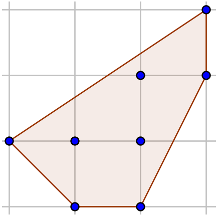

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Example

- $L(P) = 3$
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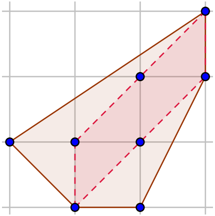

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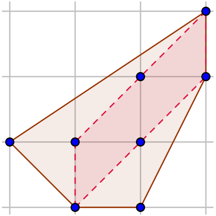
Example

- $L(P) = 3$
- Every primitive segment (i.e. with exactly two lattice points) is strongly indecomposable.



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Indeed, let $f \in \mathcal{L}_P$ be a polytope with the largest number of factors

$$f = f_1 \cdots f_L.$$

Then $P(f) = P(f_1) + \cdots + P(f_L) \subseteq P$. Hence $L \leq L(P)$.
Conversely, let $Q = Q_1 + \cdots + Q_{L(P)} \subseteq P$ be a maximal decomposition. Choose *any* g_i with $P(g_i) = Q_i$. Then the polynomial $g = g_1 \cdots g_{L(P)}$ is in $\mathcal{L}_P$ and, hence, $L(P) \leq L$.

Minkowski length $L(P)$: Properties

Simple Properties:

- Invariance: $L(P)$ is $\mathrm{AGL}(r, \mathbb{Z})$-invariant,
- Monotonicity: $L(Q) \leq L(P)$ if $Q \subseteq P$,
- Superadditivity: $L(P) + L(Q) \leq L(P + Q)$,

Note: $AGL(r, \mathbb{Z})$-equivalent polytopes produce equivalent codes!

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Some examples:

- $L(\ell \triangle_r) = \ell$ for the simplex $\triangle_r$ and any $\ell \in \mathbb{N}$.
- For $P = [0, \ell_1] \times \cdots \times [0, \ell_r]$ we have $L(P) = \ell_1 + \cdots + \ell_r$.
- $L(P \times Q) = L(P) + L(Q)$

How to compute $L(P)$?

Let $L = L(P)$. The maximal decompositions $Q_1 + \cdots + Q_L \subseteq P$ form a *poset* with respect to inclusion (up to a lattice translation). Minimal elements are smallest maximal decompositions.

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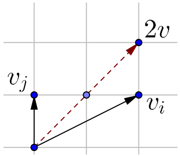
Proposition

Every smallest maximal decomposition is a sum of primitive segments with at most $2^r - 1$ *distinct direction vectors.*

Reason: The direction vectors $v_1, \ldots, v_k$ are non-zero mod $(2\mathbb{Z})^r$.
If $k \geq 2^r$ then $v_i + v_j = 2v$ for some $i < j$.



This produced a simple algorithm for computing $L(P)$ for $r = 2, 3$.

Strongly indecomposable lattice polytopes

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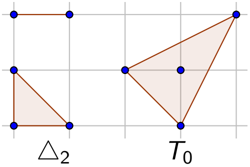
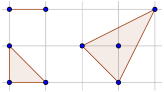
Strongly indecomposable lattice polytopes in $\mathbb{R}^2$

primitive lattice segments
two classes of triangles



A bound for toric surface codes



Note By observation and Hasse-Weil

- if $P(f) =$ primitive segment then $|Z(f)| = q - 1$
- if $P(f) = \triangle_2$ then $|Z(f)| = q - 2$
- if $P(f) = T_0$ then $|Z(f)| \leq q - 1 + 2\sqrt{q} - 1$

Moreover: Every maximal decomposition contains at most one T_0.

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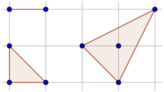

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Theorem (J. Soprunova, —, 2008)

Let P be lattice polygon in $\mathbb{R}^2$, and $q > \alpha(P)$. Then

$$d(\mathcal{C}_P) \geq (q-1)(q-1-L(P)) - (2\sqrt{q}-1)$$

(Remove $2\sqrt{q} - 1$ term if no T_0 appears in a maximal decomposition.)

Strongly indecomposable lattice polytopes in $\mathbb{R}^3$

Let $L(P) = 1$. First observations:

- P has at most $2^3 = 8$ lattice points.
- Every edge of P (in fact, every segment in P) is primitive.
- Every face of P is a triangle (either a $\triangle_2$ or a T_0).
- P can have arbitrarily large volume.

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    - *hollow and clean tetrahedra*
    - *hollow clean and non-clean double pyramids*
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- *There are* $38 + 56 + 13 = 107$ *classes of non-hollow P.*

Strongly indecomposable lattice polytopes in $\mathbb{R}^3$

What do maximal decompositions look like?

A lot has been understood recently by Beckwith, Grimm, Meyer, Soprunova, Weaver. In particular,

Theorem

- *Any maximal decomposition contains at most one polytope with more than 5 lattice points. If it does then the other summands are primitive segments.*
- *Any maximal decomposition contains at most two distinct polytopes with 4 or 5 lattice points.*

A bound for toric 3-fold codes?

What about bounds on $N_q(P)$ for strongly indecomposable P?

Theorem (Whitney, 2010)

If P belongs to a finite family then for $q > 41$

$$N_q(P) \leq 1 + F(P)/2 + (6\,\mathrm{Vol}(P) - F(P)/2 - 2)q + q^2$$

If P belongs to an infinite family then there are bounds involving parameters of the family.

No simple bound for $d(\mathcal{C}_{\mathcal{P}})$ yet, but we are hopeful!

References

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