Maximizing volume of Minkowski sum in terms of mixed volume

DMV 2020

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Main Problem

Let $K_1, \ldots, K_n$ be convex sets in $\mathbb{R}^n$ with $\mathrm{Vol}(K_i) \geq 1$ and fixed value of the mixed volume $m = V(K_1, \ldots, K_n)$. Find the sharp upper bound for the volume of the Minkowski sum in terms of m and n:

$$\mathrm{Vol}(K_1 + \cdots + K_n) \leq f(m, n).$$

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Definitions

- Minkowski addition Given $A, B \subset \mathbb{R}^n$
$$A + B = \{a + b \in \mathbb{R}^n \mid a \in A, b \in B\}.$$
- Mixed volume $V(K_1, \dots, K_n)$ is the unique *symmetric* and *multilinear* w.r.t. Minkowski addition function satisfying
$$V(K, \dots, K) = \mathrm{Vol}(K).$$

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Definitions

e.g. for $n = 2$ we have

$$\mathrm{Vol}(A + B) = V(A + B, A + B) = V(A, A) + 2V(A, B) + V(B, B)$$

Hence

$$V(A, B) = \frac{1}{2}\left(\mathrm{Vol}(A + B) - \mathrm{Vol}(A) - \mathrm{Vol}(B)\right)$$

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Plan

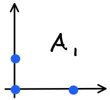
1. Motivation (sparse polynomial systems)
2. Conjecture and Results (convex geometry)
3. Methods (combinatorics)

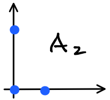
Generic polynomial systems

Example

$$\begin{cases} f_1(x,y) = c_{1,(0,0)} + c_{1,(2,0)}x^2 + c_{1,(0,1)}y = 0 \\ f_2(x,y) = c_{2,(0,0)} + c_{2,(1,0)}x + c_{2,(0,2)}y^2 = 0 \end{cases}$$

Generic $c_{1,a}, c_{2,a} \in \mathbb{C}$





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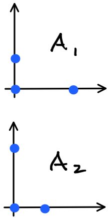

Fix finite subsets $A_1, \ldots, A_n$ of $\mathbb{Z}^n$ and pick generic coefficients

$$\{c_{i,a} \in \mathbb{C} \mid a \in A_i, 1 \le i \le n\} \setminus \text{ set of measure zero}$$

Generic Polynomial System: $f_1 = \cdots = f_n = 0$ where

$$f_i(x_1, \ldots, x_n) = \textstyle\sum_{a \in A_i} c_{i,a} x_1^{a_1} \cdots x_n^{a_n}, \quad a = (a_1, \ldots, a_n) \in A_i$$

We call A_i supports, $P_i = \mathrm{conv}(A_i)$ Newton polytopes of the system.

Esterov's result on solvability of polynomial systems

Theorem (Esterov, 2018) Let $f_1 = \cdots = f_n = 0$ be generic system with supports $A_1, \ldots, A_n$ and Newton polytopes $P_1, \ldots, P_n$ such that

- the A_i cannot be translated to a proper sublattice of $\mathbb{Z}^n$
- for any $k < n$, no k of the P_i can be translated to a k-dim subspace of $\mathbb{R}^n$

Then the system is solvable in radicals if and only if it has at most 4 solutions.

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Theorem (Bernstein–Khovanskii–Kushnirenko, 1975) A generic system with Newton polytopes $P_1, \ldots, P_n$ has exactly $n! V(P_1, \ldots, P_n)$ solutions in $(\mathbb{C} \setminus \{0\})^n$.

Two Problems of Esterov

Esterov's Problem 1 Describe all collections of lattice polytopes $(P_1, \dots, P_n)$ with a given value of their mixed volume $V(P_1, \dots, P_n)$.

Lattice equivalence Apply a simultaneous $GL(n, \mathbb{Z})$-transformation and independent lattice translations to $(P_1, \dots, P_n)$.

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Finiteness Theorem 2 (Esterov–Gusev'18) Given $n, m \in \mathbb{N}$ there are finitely many collections of n-dim'l lattice polytopes $P_1, \ldots, P_n$ in $\mathbb{R}^n$ with $n! V(P_1, \ldots, P_n) = m$, up to lattice equivalence.

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Idea: Apply the Aleksandrov-Fenchel inequality repeatedly to show

$$\mathrm{Vol}(P_1 + \cdots + P_n) \le O(m^{2^n})$$

Hence, Finiteness Theorem 1 $\implies$ Finiteness Theorem 2.

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Esterov's Question 2 What is the sharp bound for $\operatorname{Vol}(P_1 + \cdots + P_n)$?

Conjecture and Results

Let $K_1, \dots, K_n$ be convex bodies in $\mathbb{R}^n$ of volume at least 1 with $m = V(K_1, \dots, K_n)$ is fixed.

Conjecture

$$\mathrm{Vol}(K_1 + \cdots + K_n) \leq (m + n - 1)^n.$$

The above bound is attained when $K_1 = mK$ and $K_2 = \cdots = K_n = K$ with $\mathrm{Vol}(K) = 1$.

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Theorem 1 The conjecture is true for $n = 2, 3$.

Theorem 2 $\mathrm{Vol}(K_1 + \cdots + K_n) \le O(m^n)$

Estimating $\mathrm{Vol}(A+B)$ in terms of $m = V(A,B)$

$n = 2$:

Minkowski inequality: $\mathrm{Vol}(A)\,\mathrm{Vol}(B) \le V(A,B)^2$

$$\begin{aligned}\mathrm{Vol}(A+B) = V(A+B, A+B) &= V(A,A) + 2V(A,B) + V(B,B)\\ &= \mathrm{Vol}(A) + 2m + \mathrm{Vol}(B).\end{aligned}$$

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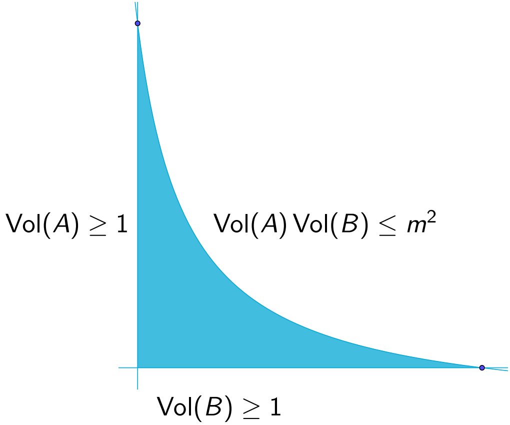

Maximum is attained when $A = mB$, $\mathrm{Vol}(B) = 1$, so

$\mathrm{Vol}(A) = m^2$ and

$\mathrm{Vol}(A+B) = (m+1)^2$

Estimating $\mathrm{Vol}(A+B+C)$ in terms of $V(A,B,C)$

$n=3$: $$\begin{aligned}\mathrm{Vol}(A+B+C) = V(A,A,A) \\ +3V(A,A,B)+3V(A,A,C) \\ +3V(A,B,B)+6V(A,B,C)+3V(A,C,C) \\ +V(B,B,B)+3V(B,B,C)+3V(B,C,C)+V(C,C,C).\end{aligned}$$

Estimating $\mathrm{Vol}(A + B + C)$ in terms of $V(A, B, C)$

$n = 3$:

$$\begin{aligned}\mathrm{Vol}(A + B + C) = {} & V(3, 0, 0) \\ & +3V(2, 1, 0) + 3V(2, 0, 1) \\ & +3V(1, 2, 0) + 6V(1, 1, 1) + 3V(1, 0, 2) \\ & +V(0, 3, 0) + 3V(0, 2, 1) + 3V(0, 1, 2) + V(0, 0, 3).\end{aligned}$$

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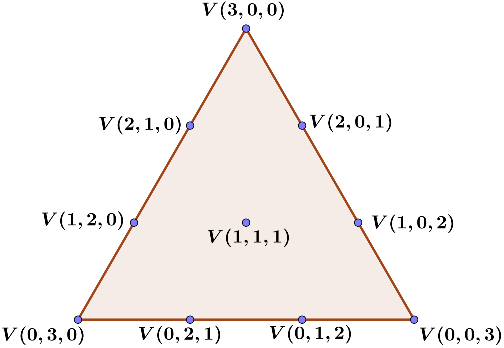

Mixed volume configuration space $\mathcal{MV}_n$

In general, for tuple $K = (K_1, \ldots, K_n)$ of convex bodies in $\mathbb{R}^n$, let

$$V_K(p) = V(\underbrace{K_1, \ldots, K_1}_{p_1}, \ldots, \underbrace{K_n, \ldots, K_n}_{p_n})$$

and $\Delta_n = \{p = (p_1, \ldots, p_n) \mid p_i \in \mathbb{Z}_{\geq 0},\ p_1 + \cdots + p_n = n\}$. Then

$$\mathrm{Vol}(K_1 + \cdots + K_n) = \sum_{p \in \Delta_n} \binom{n}{p} V_K(p).$$

We need to maximize this linear function on the mixed volume configuration space:

$$\mathcal{MV}_n = \{(V_K(p))_{p \in \Delta_n} \mid K = (K_1, \ldots, K_n) \text{ with } \mathrm{Vol}(K_i) \geq 1\}.$$

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Challenge: We know $\mathcal{MV}_n$ for $n = 2$ (Shephard'60), but not for $n > 2$.

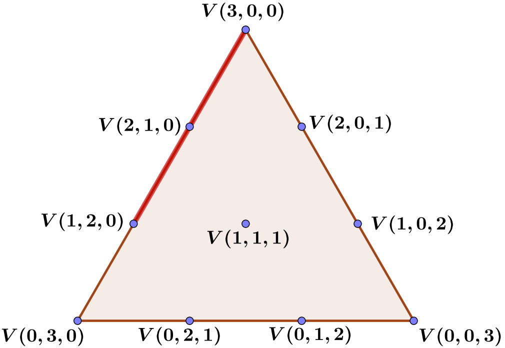
Approximating $\mathcal{MV}_n$ using Aleksandrov-Fenchel relations

Aleksandrov-Fenchel inequality

$$V(A, A, K_3, \dots, K_n)V(B, B, K_3, \dots, K_n) \le V(A, B, K_3, \dots, K_n)^2$$

These are log-concavity relations on V_K along standard directions $e_i - e_j$:

$$V_K(p + e_i - e_j)V_K(p + e_j - e_i) \le V_K(p)^2$$



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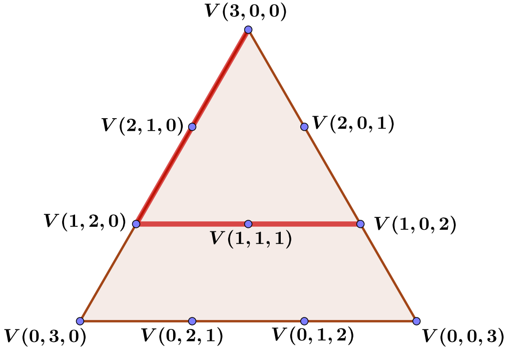

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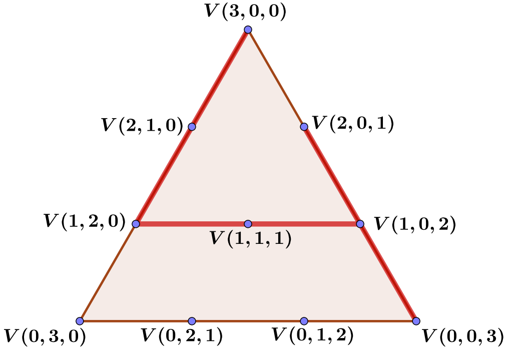

Approximating $\mathcal{MV}_n$ using Aleksandrov-Fenchel relations

We have

$$\mathcal{MV}_n \subset \mathcal{AF}_n := \{(V_p)_{p\in\Delta_n} \mid V_{p+e_i-e_j}V_{p+e_j-e_i} \leq V_p^2,\ V_p \geq 1\}.$$

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We can turn this into a linear optimization problem by taking log base m

$$\log\mathcal{MV}_n \subset \log\mathcal{AF}_n := \{(v_p)_{p\in\Delta_n} \mid v_{p+e_i-e_j} + v_{p+e_j-e_i} \leq 2v_p,\ v_p \geq 0\}.$$

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Then we can maximize the convex function in $(v_p,\ p \in \Delta_n)$

$$F := \sum_{p\in\Delta_n} \binom{n}{p} m^{v_p}$$

on the Aleksandrov-Fenchel Polytope $\mathrm{AFP}_n = \log \mathcal{AF}_n \cap \{v_{(1,\dots,1)} = 1\}$.

Approximating $\mathcal{MV}_n$ using Aleksandrov-Fenchel relations

Theorem ($n = 3$) The maximum of $\mathrm{Vol}(K_1 + K_2 + K_3)$ equals $(m + 2)^3$ where $m = V(K_1, K_2, K_3)$ and is attained when $K_1 = mK, K_2 = K_3 = K$ and $\mathrm{Vol}(K) = 1$.

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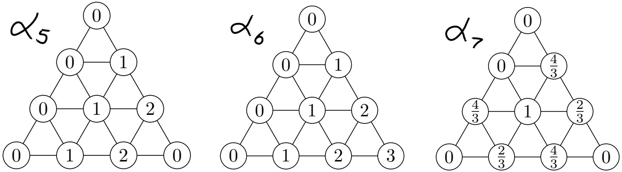
Proof Compute the 24 vertices of the A-F polytope AFP_3
(rather the seven S_3-orbits of vertices):



$$F(\alpha_6) = \sum_{p \in \Delta_n} \binom{n}{p} m^{v_p} = (1+3+3+1) + (3+6+3)m + (3+3)m^2 + m^3$$
$$= (m+2)^3$$

Approximating $\mathcal{MV}_n$ using Aleksandrov-Fenchel relations

Theorem: The Aleksandrov-Fenchel relations imply the following sharp bound

$$V_p \leq m^{|p|},$$

where $|p| = \prod_{p_i > 0} p_i$ and $m = V_{(1,\ldots,1)}$.

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Corollary: The Aleksandrov-Fenchel relations cannot produce better bound than

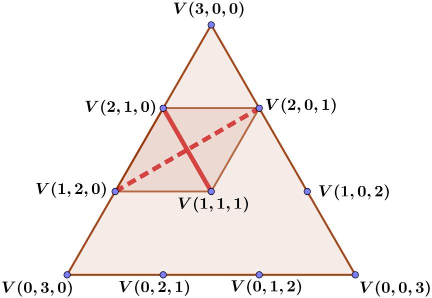
$$V(K_1 + \cdots + K_n) \leq \mathcal{O}(m^{\varepsilon(n)}),$$

where $3^{(n-2)/3} \leq \varepsilon(n) \leq 3^{n/3}$.

Approximating $\mathcal{MV}_n$ using Square relations

Square Inequality (Brazitikos, Giannopoulos, Liakopoulos '18)

$$V_K(p)V(p+a+b) \leq 2V(p+a)V(p+b), \text{ where } a = e_i - e_\ell, b = e_j - e_\ell.$$



$$\mathcal{MV}_n \subset \mathcal{SQ}_n := \{(V_p)_{p\in\Delta_n} \mid \text{A-F and Square relations}\}.$$

Approximating $\mathcal{MV}_n$ using Square relations

Theorem: The Square and Aleksandrov-Fenchel relations imply the following bound

$$V_p \leq C(n)\, m^{\max(p)},$$

where $\max(p) = \max_i(p_i)$ and $m = V_{(1,\dots,1)}$. Consequently,

$$\mathrm{Vol}(K_1 + \cdots + K_n) \leq \mathcal{O}(m^n).$$

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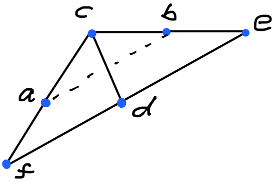
By the way, $C(n) = 2^{n(n-1)\binom{\lfloor n/2 \rfloor}{2}}$

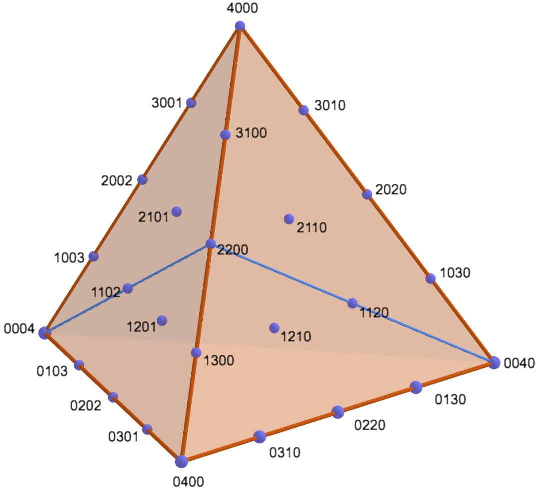
Approximating $\mathcal{MV}_n$ using Square relations

Square and Aleksandrov-Fenchel inequalities combined produce new (weak) log-concavity directions!



$$\begin{cases} (ab)^2 \le (2cd)^2 \\ ce \le b^2 \\ cf \le a^2 \end{cases} \quad \Rightarrow \quad ef \le 4d^2$$

Approximating $\mathcal{MV}_n$ using Square relations



How to show that $V_{2200} \leq 4m^2$?

Multiply two new log-concave directions and one (A-F)^2:

$V_{2200} \leq 4V_{1120}^2$
$V_{2200} \leq 4V_{1102}^2$
$(V_{1120}V_{1102})^2 \leq (m^2)^2$

$\Rightarrow V_{2200} \leq 4m^2$

Reference

- Gennadiy Averkov, Christopher Borger, and Ivan Soprunov, Inequalities between mixed volumes of convex bodies: volume bounds for the Minkowski sum, Mathematika 66 (2020) 1003–1027, arXiv e-prints (2020), arXiv:2002.03065.

Thank you!