Strict monotonicity of the mixed volume

Fall 2018 AMS meeting, Ann Arbor

Ivan Soprunov
(with Frederic Bihan)

Cleveland State University

October 19, 2018

Sparse Polynomial Systems

Study solutions to Laurent polynomial systems in the torus $(\mathbb{C}^*)^n$.
Sparse Polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$:

$$f = \sum_{a \in \mathcal{A}} c_a x^a, \text{ where } x^a = x_1^{a_1} \cdots x_n^{a_n}, \quad c_a \in \mathbb{C}^*.$$

The set of exponents $\mathcal{A} \subset \mathbb{Z}^n$ is the support of f.
Its convex hull $P = \operatorname{conv}(\mathcal{A})$ is the Newton Polytope of f.

Sparse Polynomial Systems

Study solutions to Laurent polynomial systems in the torus $(\mathbb{C}^*)^n$.
Sparse Polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$:

$$f = \sum_{a \in \mathcal{A}} c_a x^a, \text{ where } x^a = x_1^{a_1} \cdots x_n^{a_n}, \quad c_a \in \mathbb{C}^*.$$

The set of exponents $\mathcal{A} \subset \mathbb{Z}^n$ is the support of f.
Its convex hull $P = \mathrm{conv}(\mathcal{A})$ is the Newton Polytope of f.

Sparse Polynomial System

$$\begin{cases} f_1 = 0, \text{ support } \mathcal{A}_1 \\ \quad \cdots \\ f_n = 0, \text{ support } \mathcal{A}_n \end{cases}$$

In matrix form:

$$Cx^{\mathcal{A}} = 0, \qquad x^{\mathcal{A}} = \begin{pmatrix} x^{a_1} \\ \vdots \\ x^{a_N} \end{pmatrix}$$

where $\mathcal{A} = \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_n$ total support, $\mathcal{A} = \{a_1, \ldots, a_N\} \subset \mathbb{Z}^n$
and $C \in M_{n \times N}(\mathbb{C})$ coefficient matrix

BKK Bound

Theorem (Kushnirenko 1976)

The system $Cx^{\mathcal{A}} = 0$ *has at most* $n! \operatorname{vol}(P)$ *isolated solutions in* $(\mathbb{C}^*)^n$*, where* $P = \operatorname{conv}(\mathcal{A})$ *is the Newton polytope of the system.*

BKK Bound

Theorem (Kushnirenko 1976)

The system $Cx^{\mathcal{A}} = 0$ *has at most* $n!\,\mathrm{vol}(P)$ *isolated solutions in* $(\mathbb{C}^*)^n$*, where* $P = \mathrm{conv}(\mathcal{A})$ *is the Newton polytope of the system.*

Theorem (Bernstein–Kushnirenko–Khovanskii 1978)

The system $Cx^{\mathcal{A}} = 0$ *has at most* $n!v(P_1, \ldots, P_n)$ *isolated solutions in* $(\mathbb{C}^*)^n$*, where* $P_i = \mathrm{conv}(\mathcal{A}_i)$ *is the Newton polytope of the* f_i*.*

Here $v(P_1, \ldots, P_n)$ is the mixed volume of the polytopes $P_1, \ldots, P_n$.

Moreover, the bounds are met iff certain "facial subsystems" are inconsistent.

BKK Bound

Theorem (Kushnirenko 1976)

The system $Cx^{\mathcal{A}} = 0$ has at most $n! \operatorname{vol}(P)$ isolated solutions in $(\mathbb{C}^*)^n$, where $P = \operatorname{conv}(\mathcal{A})$ is the Newton polytope of the system.

Theorem (Bernstein–Kushnirenko–Khovanskii 1978)

The system $Cx^{\mathcal{A}} = 0$ has at most $n! v(P_1, \ldots, P_n)$ isolated solutions in $(\mathbb{C}^*)^n$, where $P_i = \operatorname{conv}(\mathcal{A}_i)$ is the Newton polytope of the f_i.

Here $v(P_1, \ldots, P_n)$ is the mixed volume of the polytopes $P_1, \ldots, P_n$.

Moreover, the bounds are met iff certain "facial subsystems" are inconsistent.

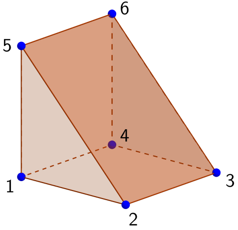
Question: Is there a quick way to see if the bound $n! \operatorname{vol}(P)$ is met without checking inconsistency of facial subsystems?

BKK Bound

Answer: Yes, sometimes.
Example: Consider the system

$$\begin{cases} f_1 = 1 + 3x + 5xy + \ y - 2z + 2yz = 0, \\ f_2 = 1 + \ x - 3xy + 3y + \ z - \ yz = 0, \\ f_3 = 1 + 3x + \ xy + 3y - \ z + \ yz = 0. \end{cases}$$

$P_1 = P_2 = P_3 = P$

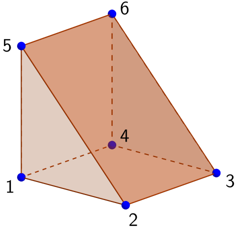


BKK Bound

Answer: Yes, sometimes.
Example: Consider the system

$$\begin{cases} f_1 = 1 + 3x + 5xy + \ y - 2z + 2yz = 0, \\ f_2 = 1 + \ x - 3xy + 3y + \ z - \ yz = 0, \\ f_3 = 1 + 3x + \ xy + 3y - \ z + \ yz = 0. \end{cases}$$

$P_1 = P_2 = P_3 = P$



By Kushnirenko bound it has at most $3!\,\mathrm{vol}(P) = 3$ isolated solutions.
In fact, it has less! Here is how we can see that.

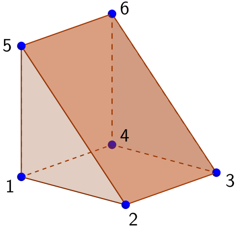
BKK Bound

Example: Consider the system $Cx^{\mathcal{A}} = 0$.

$$C = \begin{pmatrix} 1 & 3 & 5 & 1 & -2 & 2 \\ 1 & 1 & -3 & 3 & 1 & -1 \\ 1 & 3 & 1 & 3 & -1 & 1 \end{pmatrix}$$

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$P_1 = P_2 = P_3 = P$



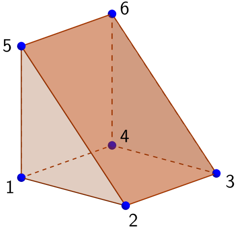
BKK Bound

Example: Consider the system $Cx^{\mathcal{A}} = 0$.

$$C = \begin{pmatrix} 1 & 3 & 5 & 1 & -2 & 2 \\ 1 & 1 & -3 & 3 & 1 & -1 \\ 1 & 3 & 1 & 3 & -1 & 1 \end{pmatrix}$$

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$P_1 = P_2 = P_3 = P$



Theorem (Bihan, S, 2018): Check if $\operatorname{rk} C_F \geq \operatorname{rk} \bar{A}_F$ for every face $F \subsetneq P$. If not, the bound $n! \operatorname{vol}(P)$ is not met.

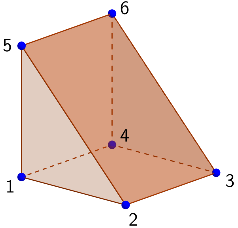
BKK Bound

Example: Consider the system $Cx^{\mathcal{A}} = 0$.

$$C = \begin{pmatrix} 1 & 3 & 5 & 1 & -2 & 2 \\ 1 & 1 & -3 & 3 & 1 & -1 \\ 1 & 3 & 1 & 3 & -1 & 1 \end{pmatrix}$$

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$P_1 = P_2 = P_3 = P$



Theorem (Bihan, S, 2018): Check if $\operatorname{rk} C_F \geq \operatorname{rk} \bar{A}_F$ for every face $F \subsetneq P$.
If not, the bound $n! \operatorname{vol}(P)$ is not met.

Here $\operatorname{rk} C_F < \operatorname{rk} \bar{A}_F$ for $F = \{5, 6\}$.

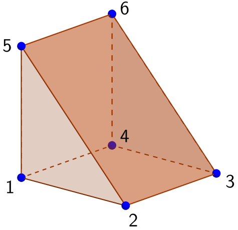
BKK Bound

Example: Consider the system $Cx^{\mathcal{A}} = 0$.

$$C = \begin{pmatrix} 1 & 3 & 5 & 1 & -2 & 2 \\ 1 & 1 & -3 & 3 & 1 & -1 \\ 1 & 3 & 1 & 3 & -1 & 1 \end{pmatrix}$$

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$P_1 = P_2 = P_3 = P$



Theorem (Bihan, S, 2018): Check if $\operatorname{rk} C_F \geq \operatorname{rk} \bar{A}_F$ for every face $F \subsetneq P$.
If not, the bound $n! \operatorname{vol}(P)$ is not met.

Here $\operatorname{rk} C_F < \operatorname{rk} \bar{A}_F$ for $F = \{5, 6\}$.

Reason: $Cx^{\mathcal{A}} = 0$ is equivalent to a system with $v(P_1, P_2, P_3) < \operatorname{vol}(P)$.

Mixed Volume: Definition

Recall the Minkowski sum $P+Q=\{p+q\in\mathbb{R}^n \mid p\in P, q\in Q\}$ for any $P,Q\subset\mathbb{R}^n$.

Mixed Volume is the coefficient of $\lambda_1\cdots\lambda_n$ in the polynomial

$$\mathrm{vol}(\lambda_1P_1+\cdots+\lambda_nP_n)=\mathrm{vol}(P_1)\lambda_1^n+\cdots+v(P_1,\dots,P_n)\lambda_1\cdots\lambda_n+\dots$$

It can be expressed as

$$v(P_1,\dots,P_n)=\frac{1}{n!}\sum_{m=1}^{n}(-1)^{n+m}\sum_{i_1<\dots<i_m}\mathrm{vol}_n(P_{i_1}+\cdots+P_{i_m})$$

Mixed Volume: Definition

Recall the Minkowski sum $P+Q=\{p+q\in\mathbb{R}^n \mid p\in P, q\in Q\}$ for any $P,Q\subset\mathbb{R}^n$.

Mixed Volume is the coefficient of $\lambda_1\cdots\lambda_n$ in the polynomial

$$\mathrm{vol}(\lambda_1 P_1+\cdots+\lambda_n P_n)=\mathrm{vol}(P_1)\lambda_1^n+\cdots+v(P_1,\dots,P_n)\lambda_1\cdots\lambda_n+\dots$$

It can be expressed as

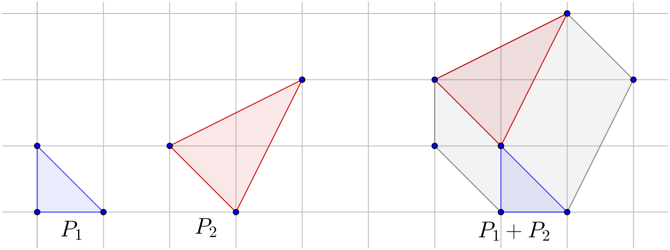
$$v(P_1,\dots,P_n)=\frac{1}{n!}\sum_{m=1}^{n}(-1)^{n+m}\sum_{i_1<\cdots<i_m}\mathrm{vol}_n(P_{i_1}+\cdots+P_{i_m})$$

Note: Mixed volume is symmetric, additive, and satisfies

$$v(P,\dots,P)=\mathrm{vol}(P).$$

Mixed Volume: Example

Example: Consider P_1, P_2 in $\mathbb{R}^2$



We have $v(P_1, P_2) = \frac{1}{2}\left(\text{vol}(P_1 + P_2) - \text{vol}(P_1) - \text{vol}(P_2)\right) = 2.$

Mixed Volume: Properties

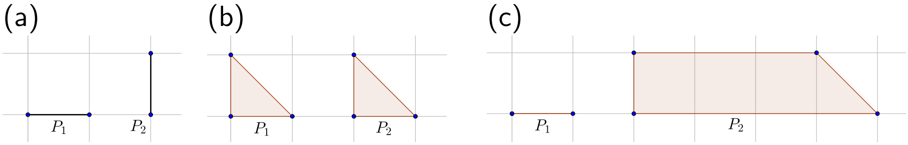
Non-negativity and Monotonicity:

- $v(P_1, \ldots, P_n) \geq 0$
- $v(P_1, \ldots, P_n) \leq v(Q_1, \ldots, Q_n)$ for $P_i \subseteq Q_i$, $1 \leq i \leq n$.

Mixed Volume: Properties

Non-negativity and Monotonicity:

- $v(P_1, \ldots, P_n) \geq 0$
- $v(P_1, \ldots, P_n) \leq v(Q_1, \ldots, Q_n)$ for $P_i \subseteq Q_i$, $1 \leq i \leq n$.



In each case $v(P_1, P_2) = \frac{1}{2}$.

Mixed Volume: Strict Positivity

Definition: A collection $(P_1, \ldots, P_n)$ for is called non-degenerate if there exists segments $S_i \subseteq P_i$ with linearly independent directions.

Mixed Volume: Strict Positivity

Definition: A collection $(P_1, \ldots, P_n)$ for is called non-degenerate if there exists segments $S_i \subseteq P_i$ with linearly independent directions.

Theorem (Minkowski)

Let $P_1, \ldots, P_n$ be convex bodies in $\mathbb{R}^n$. Then $v(P_1, \ldots, P_n) > 0$ iff $(P_1, \ldots, P_n)$ is non-degenerate.

Mixed Volume: Strict Monotonicity

Special Case:
When is $v(P_1, P_2, \ldots, P_n) < v(Q_1, P_2, \ldots, P_n)$ for $P_1 \subseteq Q_1$?

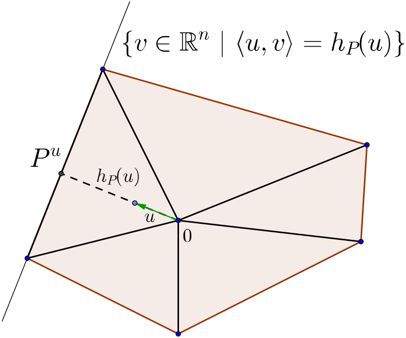
Mixed Volume: Strict Monotonicity

Special Case:
When is $v(P_1, P_2, \dots, P_n) < v(Q_1, P_2, \dots, P_n)$ for $P_1 \subseteq Q_1$?

Let $h_P : \mathbb{S}^{n-1} \to \mathbb{R}$, $h_P(u) = \max\{\langle u, v\rangle \mid v \in P\}$ be the support function of P and $P^u = P \cap \{v \in \mathbb{R}^n \mid \langle u, v\rangle = h_P(u)\}$ the face of P corresponding to u.

Integral Formula:

$$\mathrm{vol}(P) = \frac{1}{n} \sum_{u \in \mathbb{S}^{n-1}} h_P(u)\,\mathrm{vol}(P^u)$$



Mixed Volume: Strict Monotonicity

Special Case:

When is $v(P_1, P_2, \ldots, P_n) < v(Q_1, P_2, \ldots, P_n)$ for $P_1 \subseteq Q_1$?

Integral Formula:

$$v(P_1, \ldots, P_n) = \frac{1}{n} \sum_{u \in \mathbb{S}^{n-1}} h_{P_1}(u) v(P_2^u, \ldots, P_n^u)$$

Note:

- $h_{P_1}(u) \leq h_{Q_1}(u)$ for all $u \in \mathbb{S}^{n-1}$ if and only if $P_1 \subseteq Q_1$
- $h_{P_1}(u) < h_{Q_1}(u)$ if and only if P_1^u does not touch Q_1^u

Mixed Volume: Strict Monotonicity

Special Case:
When is $v(P_1, P_2, \ldots, P_n) < v(Q_1, P_2, \ldots, P_n)$ for $P_1 \subseteq Q_1$?

Integral Formula:

$$v(P_1, \ldots, P_n) = \frac{1}{n} \sum_{u \in \mathbb{S}^{n-1}} h_{P_1}(u) v(P_2^u, \ldots, P_n^u)$$

Note:

- $h_{P_1}(u) \leq h_{Q_1}(u)$ for all $u \in \mathbb{S}^{n-1}$ if and only if $P_1 \subseteq Q_1$
- $h_{P_1}(u) < h_{Q_1}(u)$ if and only if P_1^u does not touch Q_1^u

Therefore, $v(P_1, P_2, \ldots, P_n) < v(Q_1, P_2, \ldots, P_n)$ if and only if there exists $u \in \mathbb{S}^{n-1}$ such that

- P_1^u does not touch Q_1^u and
- $(P_2^u, \ldots, P_n^u)$ is non-degenerate.

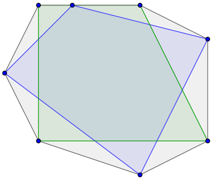
Mixed Volume: Strict Monotonicity

Theorem (Bihan-S, 2018)

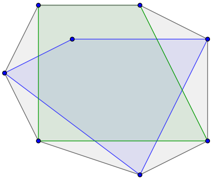
Let $P_1, \ldots, P_n$ be polytopes in $\mathbb{R}^n$ contained in an n-dimensional polytope P. Then

$$v(P_1, \ldots, P_n) < \mathrm{vol}(P)$$

if and only if there is a proper face of P of dimension k which is touched by at most k of the $P_1, \ldots, P_n$.



$v(P_1, P_2) = v(P)$



$v(P_1, P_2) < v(P)$

P_1 blue
P_2 green
P gray

Application to sparse systems

Corollary (Bihan-S, 2018)

Let $Cx^{\mathcal{A}} = 0$ *be a sparse system with Newton polytope* $P = \mathrm{conv}(\mathcal{A})$*. If there exists a face* $F \subset P$ *such that* $\mathrm{rk}\, C_F < \mathrm{rk}\, \bar{A}_F$ *then the system has less than* $n!\,\mathrm{vol}(P)$ *isolated solutions.*

Application to sparse systems

Corollary (Bihan-S, 2018)

Let $Cx^{\mathcal{A}} = 0$ be a sparse system with Newton polytope $P = \operatorname{conv}(\mathcal{A})$. If there exists a face $F \subset P$ such that $\operatorname{rk} C_F < \operatorname{rk} \bar{A}_F$ then the system has less than $n! \operatorname{vol}(P)$ isolated solutions.

Corollary (Bihan-S, 2018)

Let $Cx^{\mathcal{A}} = 0$ be a sparse system with n-dim Newton polytope $P = \operatorname{conv}(\mathcal{A})$. If no maximal minor of C vanishes then the system has exactly $n! \operatorname{vol}(P)$ isolated solutions.

Mixed Volume: Strict Monotonicity

Theorem (Bihan-S, 2018)

Let $P_1 \subseteq Q_1, \ldots, P_n \subseteq Q_n$ be polytopes in $\mathbb{R}^n$. Then

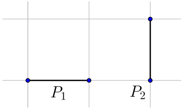
$$v(P_1, \ldots, P_n) < v(Q_1, \ldots, Q_n)$$

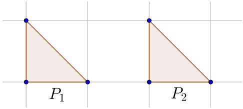
if and only if there is $u \in \mathbb{S}^{n-1}$ such that the collection

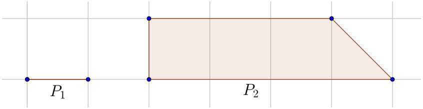
$$\underbrace{(Q_i^u \mid P_i^u \subseteq Q_i^u)}_{\text{touched faces}} \cup (Q_i \mid P_i^u \not\subseteq Q_i^u)$$

is non-degenerate.

Questions

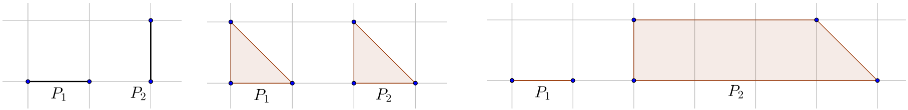






1. Can one describe minimal/maximal collections $(P_1, \dots, P_n)$ with fixed $v(P_1, \dots, P_n)$?

Questions



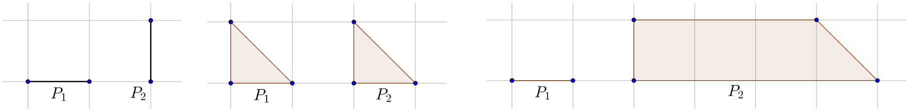
1. Can one describe minimal/maximal collections $(P_1, \ldots, P_n)$ with fixed $v(P_1, \ldots, P_n)$?

2. If $v(P_1, \ldots, P_n) < \mathrm{vol}(P)$ then

$$n!\,\mathrm{vol}(P) - n!\,v(P_1, \ldots, P_n) \geq 1.$$

Can one have a better estimate for this gap?

Questions



1. Can one describe minimal/maximal collections $(P_1, \dots, P_n)$ with fixed $v(P_1, \dots, P_n)$?

2. If $v(P_1, \dots, P_n) < \mathrm{vol}(P)$ then

$$n!\,\mathrm{vol}(P) - n!\,v(P_1, \dots, P_n) \geq 1.$$

Can one have a better estimate for this gap?

— The End —