Minkowski Length of Lattice Polytopes

Einstein Workshop on Lattice Polytopes

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December 12, 2016

Counting rational points on hypersurfaces

Tsfasman's Question (1989, Luminy): What is the largest number of $\mathbb{F}_q$-points on a hypersurface $H \subset \mathbb{P}^d$ of degree t? That is,

$$N_q(t,d) = \max_{\deg f = t} \{p \in \mathbb{P}^d(\mathbb{F}_q) \mid f(p) = 0\},$$

over all homogeneous $f \in \mathbb{F}_q[x_0, \dots, x_d]$ of degree t?

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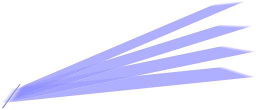
Serre's Answer (1989): For $q \geq t$, the polynomials that factor the most have the most zeroes over $\mathbb{F}_q$. Thus we should take f to be a product of linear factors and so

$$N_q(t,d) = t\frac{q^d - 1}{q-1} - (t-1)\frac{q^{d-1}-1}{q-1} = tq^{d-1} + q^{d-2} + \cdots + 1.$$



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More generally, let X be a toric variety and D is a $\mathbb{T}$-invariant Cartier divisor on X with polytope $P = P_D$. Then the $\mathbb{F}_q$-global sections of $\mathcal{O}_X(D)$ can be identified with

$$\mathcal{L}(P) := \bigoplus_{a \in P \cap \mathbb{Z}^d} \mathbb{F}_q x^a.$$

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Question: Given P, what is the largest number of zeroes $N_q(P)$ in $(\mathbb{F}_q^*)^d$ a polynomial $f \in \mathcal{L}(P)$ may have?

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Easier Questions:

- What is the largest number of factors a polynomial $f \in \mathcal{L}(P)$ may have?
- What do the factors in this case look like?

Counting rational points on hypersurfaces

In fact, the easier questions are about the geometry of P.

- The largest number of factors $f \in \mathcal{L}(P)$ may have is the Minkowski length of P.
- The irreducible factors of such f have Newton polytopes that are strongly indecomposable.

Minkowski length: Definition

Let P be a lattice polytope in $\mathbb{R}^d$.

Definition: The largest number of lattice polytopes of positive dimension whose Minkowski sum is contained in P is called the Minkowski length:

$$L(P) = \max\{L \in \mathbb{N} \mid Q = Q_1 + \cdots + Q_L \subseteq P, \dim Q_i > 0\}.$$

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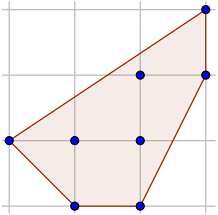
Example

- $L(P) = 3$
- $L(t\Delta) = t$ for a unimodular simplex Δ and $t \in \mathbb{N}$.



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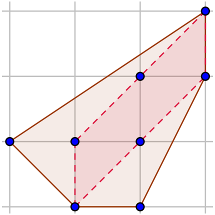

Minkowski length: Properties

Simple Properties:

- Invariance: $L(P)$ is $AGL(d, \mathbb{Z})$-invariant,
- Monotonicity: $L(Q) \leq L(P)$ if $Q \subseteq P$,
- Superadditivity: $L(P) + L(Q) \leq L(P + Q)$,

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- Superadditivity: $L(P) + L(Q) \leq L(P + Q)$,
- Bound:

$$|P \cap \mathbb{Z}^d| \leq (L(P) + 1)^d$$

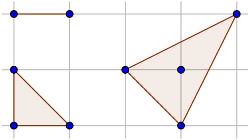
In particular, if P is strongly indecomposable then

$$|P \cap \mathbb{Z}^d| \leq 2^d.$$

...Think about $P \cap (\mathbb{Z}/2\mathbb{Z})^d$...

Strongly indecomposable lattice polytopes in small dimension

dim $P = 1$ primitive lattice segments
dim $P = 2$ two classes of triangles



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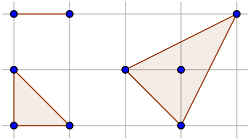
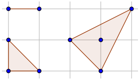


Theorem (Josh Whitney, 2010)

Let P be strongly indecomposable, $\dim P = 3$. Then

- *P may have 4, 5, or 6 vertices only*
- *There are infinite families of such P:*
  - *empty and clean tetrahedra*
  - *empty clean and non-clean double pyramids*
  - *empty clean and non-clean 6 vertex polytopes*
- *There are $38 + 56 + 13 = 107$ classes of non-empty P*

Back to counting rational points on hypersurfaces



Theorem (S-Soprunova, '08)

Let P be a lattice polygon with $L = L(P)$. Then

$$N_q(P) \leq L(q-1) + 2\sqrt{q} - 1$$

for $q > \alpha(P)$.

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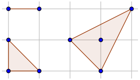

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Let P be a lattice polytope in $\mathbb{R}^3$ with $L = L(P)$. Then for $q > \beta(P)$

$$N_q(P) \le N_q(Q_1) + \cdots + N_q(Q_L),$$

for any $Q_1 + \cdots + Q_L \subseteq P$ and $N_q(Q_i)$ have explicit formulas based on the classification of 3-dim strongly indecomposable polytopes.

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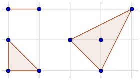

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Theorem (Beckwith–Grimm–Soprunova–Weaver, '12)

The total number of interior points in the Q_i in any maximal decomposition is at most 4.

How to compute $L(P)$?

Let $L = L(P)$. The maximal decompositions $Q_1 + \cdots + Q_L \subseteq P$ form a poset with respect to inclusion (up to a lattice translation). Minimal elements are smallest maximal decompositions.

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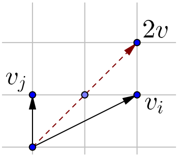
Proposition

Every smallest maximal decomposition is a lattice zonotope $Z_{\min} \subseteq P$ with at most $2^d - 1$ distinct direction vectors.

Reason: The direction vectors $v_1, \ldots, v_k$ are non-zero mod 2. If $k \geq 2^d$ then $v_i + v_j = 2v$ for some $i < j$.



Relation to Lattice Diameter

Let P be a lattice polytope in $\mathbb{R}^d$.

Definition: The largest number of lattice polytopes of positive dimension whose Minkowski sum is at most n-dimensional and is contained in P is called the n-th Minkowski length:

$$L_n(P) = \max\{L \in \mathbb{N} \mid Q = Q_1 + \cdots + Q_L \subseteq P, \dim Q_i > 0, \dim Q \leq n\}.$$

Clearly $L_1(P) \leq L_2(P) \leq \cdots \leq L_d(P) = L(P)$.
Note that $L_1(P) =$ lattice diameter of P.

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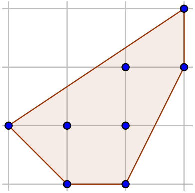
Example

- $L_1(P) = 2$, $L_2(P) = 3$
- $L_n(t\Delta) = t$ for any $n \in \mathbb{N}$



Rational Minkowski length

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Definition: Define rational (asymptotic) Minkowski length:

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In particular, $\lambda_1(P)$ is the rational diameter, that is

$$\lambda_1(P) = \max\{s_P(v) \mid \text{primitive } v \in \mathbb{Z}^d\},$$

where $s_P(v)$ is the diameter of P in the direction of v (relative to $v\mathbb{Z} \subset \mathbb{Z}^d$). This implies

$$L_1(tP) = \lfloor \lambda_1(tP) \rfloor = \lfloor \lambda_1(P)t \rfloor,$$

which is quasi-linear (i.e. quasi-polynomial in t with linear constituencies).

Rational Minkowski length

Theorem (S-Soprunova '16)

Let P be a lattice polytope in $\mathbb{R}^d$. There exists $k \in \mathbb{N}$ such that

$$\lambda(P) = \frac{L(kP)}{k}.$$

The smallest such k is the period of P.

Corollary: $\lambda(tP) = t\lambda(P)$ for any $t \in \mathbb{N}$.

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Sketch of the proof:

- $\lambda(P)$ equals the supremum of the "normalized perimeter" $p(Z)$ of all rational zonotopes $Z \subseteq P$.
- The number of directions for the summands of $Z_{\min}$ is bounded by $2^d - 1$

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Sketch of the proof:

- There are only finitely many collections of directions for $Z_{\min}$ for which $p(Z)$ is "close" to $\lambda(P)$
- $p(Z)$ is the maximum of a linear function on a finite set of rational polytopes

Eventual Quasi-linearity of Minkowski length

Theorem (S-Soprunova '16)

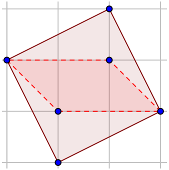
Let P be a lattice polytope in $\mathbb{R}^d$ with period k. Then $L(tP)$ is eventually quasi-linear in t, that is, there exist $c_r \in \mathbb{Z}$ for $0 \le r < k$ such that for all $t \gg 0$

$$L(tP) = k\lambda(P)\left\lfloor \frac{t}{k} \right\rfloor + c_r, \text{ whenever } t \equiv r \bmod k.$$

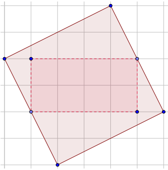
Moreover, $L(rP) \le c_r \le r\lambda(P)$.

Remark: The same statement holds for $L_n(tP)$ for any $n \in \mathbb{N}$.

Example



P



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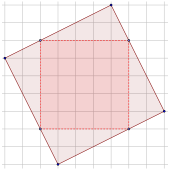

$3P$

1-st Minkowski length (lattice diameter)

- We have $L_1(P) = 2$ and $\lambda_1(P) = \frac{5}{2}$
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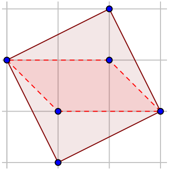

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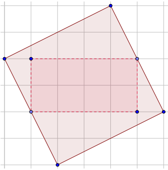

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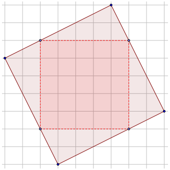

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2-nd Minkowski length

- We have $\lambda(P) \le \frac{2Vol_2(P)}{w(P)}$ for any polygon P
- Here $\lambda(P) = \frac{10}{3}$ and P has period $k = 3$.
- $L(tP) = 10\lfloor \frac{t}{3} \rfloor + \{0,3,6\}$

Questions

1. As we saw earlier $L_n(t\Delta) = L_1(t\Delta)$ for any $n \in \mathbb{N}$. It turns out that $L_n(T) = L_1(T)$ for any triangle in $\mathbb{R}^2$. Does this hold for simplices in any dimension?

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