Bezout inequality for mixed volumes

AMS Meeting, Fargo ND

Ivan Soprunov
(with Christos Saroglou and Artem Zvavitch)

Cleveland State University

April 15, 2016

Motivation: Bezout's Theorem

Hypersurface in $\mathbb{C}P^n$

$X = \{x \in \mathbb{C}P^n \mid F(x) = 0\}$, where F a is homogeneous polynomial

$\deg X = \deg F =$ number if intersections of X with generic line

Motivation: Bezout's Theorem

Hypersurface in $\mathbb{C}P^n$

$X = \{x \in \mathbb{C}P^n \mid F(x) = 0\}$, where F a is homogeneous polynomial

$\deg X = \deg F =$ number if intersections of X with generic line

Intersection of hypersurfaces

$X_1, \dots, X_r$ hypersurfaces in $\mathbb{C}P^n$

$\deg(X_1 \cap \dots \cap X_r) =$ number of intersections of $X_1 \cap \dots \cap X_r$ with a generic subspace of dimension r

Motivation: Bezout's Theorem

Hypersurface in $\mathbb{C}P^n$

$X = \{x \in \mathbb{C}P^n \mid F(x) = 0\}$, where F a is homogeneous polynomial

$\deg X = \deg F =$ number if intersections of X with generic line

Intersection of hypersurfaces

$X_1, \ldots, X_r$ hypersurfaces in $\mathbb{C}P^n$

$\deg(X_1 \cap \cdots \cap X_r) =$ number of intersections of $X_1 \cap \cdots \cap X_r$ with a generic subspace of dimension r

Theorem (Bezout)

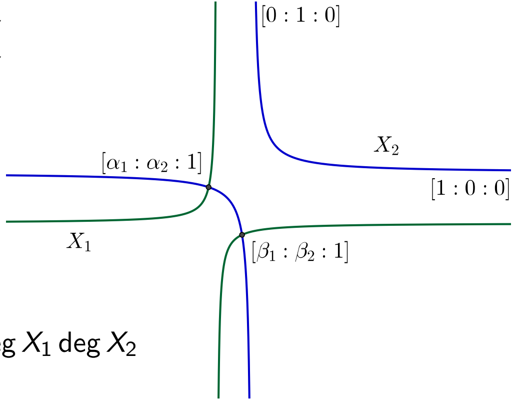
$$\deg(X_1 \cap \cdots \cap X_r) = \prod_{i=1}^{r} \deg X_i.$$

Small Example ($n = r = 2$)

$$F_1 = a_0 z^2 + a_1 xz + a_2 yz + a_3 xy$$
$$F_2 = b_0 z^2 + b_1 xz + b_2 yz + b_3 xy$$

$\deg X_1 = \deg X_2 = 2$

$\deg(X_1 \cap X_2) = 4$



Bezout: $\deg(X_1 \cap X_2) = \deg X_1 \deg X_2$

Bernstein-Kushnirenko-Khovanskii theorem

Newton Polytope

$NP(f) =$ convex hull of exponent vectors of a polynomial f

Theorem (BKK)

Let $f_1, \ldots, f_n$ be polynomials with fixed NP's $P_1, \ldots, P_n \subset \mathbb{R}^n$ and generic coefficients. Then

$$\#\{x \in (\mathbb{C}^*)^n \mid f_1(x) = \cdots = f_n(x) = 0\} = n! V(P_1, \ldots, P_n).$$

Bernstein-Kushnirenko-Khovanskii theorem

Newton Polytope

$NP(f) =$ convex hull of exponent vectors of a polynomial f

Theorem (BKK)

Let $f_1, \ldots, f_n$ be polynomials with fixed NP's $P_1, \ldots, P_n \subset \mathbb{R}^n$ and generic coefficients. Then

$$\#\{x \in (\mathbb{C}^*)^n \mid f_1(x) = \cdots = f_n(x) = 0\} = n! V(P_1, \ldots, P_n).$$

Corollary Let $\Delta = \operatorname{conv}\{0, e_1, \ldots, e_n\}$ be the standard n-simplex.

- $$\begin{aligned} \deg(X_1 \cap \cdots \cap X_r) \geq \#\{x \in (\mathbb{C}^*)^n \mid\ & f_1(x) = \cdots = f_r(x) = 0, \\ & \ell_1(x) = \cdots = \ell_{n-r}(x) = 0\} \\ = n! V(P_1, \ldots, & P_r, \Delta^{n-r}), \end{aligned}$$

Bernstein-Kushnirenko-Khovanskii theorem

Newton Polytope

$NP(f) =$ convex hull of exponent vectors of a polynomial f

Theorem (BKK)

Let $f_1, \dots, f_n$ be polynomials with fixed NP's $P_1, \dots, P_n \subset \mathbb{R}^n$ and generic coefficients. Then

$$\#\{x \in (\mathbb{C}^*)^n \mid f_1(x) = \cdots = f_n(x) = 0\} = n! V(P_1, \dots, P_n).$$

Corollary Let $\Delta = \mathrm{conv}\{0, e_1, \dots, e_n\}$ be the standard n-simplex.

- $$\begin{aligned}\deg(X_1 \cap \cdots \cap X_r) \geq \#\{x \in (\mathbb{C}^*)^n \mid\ & f_1(x) = \cdots = f_r(x) = 0, \\ & \ell_1(x) = \cdots = \ell_{n-r}(x) = 0\} \\ = n! V(P_1, \dots, P_r, \Delta^{n-r}),&\end{aligned}$$
- Suppose the NP's touch the coordinate hyperplanes. Then

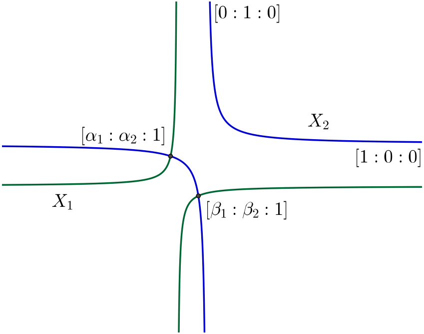
$$\deg(X_i) = n! V(P_i, \Delta^{n-1}).$$

Small Example ($n = r = 2$)

$$f_1 = a_0 + a_1 x + a_2 y + a_3 xy$$
$$f_2 = b_0 + b_1 x + b_2 y + b_3 xy$$

$$P_1 = P_2 = \square$$

$$\deg X_i = 2!V(\square, \Delta) = 2$$
$$\deg(X_1 \cap X_2) \geq 2!V(\square, \square) = 2$$



Bezout: $2!V(\square, \square) \leq 2!V(\square, \Delta) 2!V(\square, \Delta)$

Bezout Inequality for Mixed Volumes

In general, we have

$$n!V(P_1,\dots,P_r,\Delta^{n-r}) \le \prod_{i=1}^{r} n!V(P_i,\Delta^{n-1}).$$

Since $V_n(\Delta) = 1/n!$ this is equivalent to

$$V_n(\Delta)^{r-1}V(P_1,\dots,P_r,\Delta^{n-r}) \le \prod_{i=1}^{r} V(P_i,\Delta^{n-1}).$$

Bezout Inequality for Mixed Volumes

Theorem

For any convex bodies $P_1, \dots, P_r$ and any n-simplex Δ in $\mathbb{R}^n$

$$\text{(BMV-r)}\ V_n(\Delta)^{r-1} V(P_1, \dots, P_r, \Delta^{n-r}) \le \prod_{i=1}^{r} V(P_i, \Delta^{n-1}).$$

Bezout Inequality for Mixed Volumes

Theorem

For any convex bodies $P_1, \ldots, P_r$ and any n-simplex Δ in $\mathbb{R}^n$

$$\textit{(BMV-r)}\quad V_n(\Delta)^{r-1} V(P_1, \ldots, P_r, \Delta^{n-r}) \leq \prod_{i=1}^{r} V(P_i, \Delta^{n-1}).$$

Proof.

Rescale & translate $P_1, \ldots, P_r$ such that each P_i is inscribed in Δ. Then

- $V(P_i, \Delta^{n-1}) = V_n(\Delta)$,
- $V(P_1, \ldots, P_r, \Delta^{n-r}) \leq V_n(\Delta)$ by monotonicity.

□

Conjecture

The special, but the most important case is when $r = 2$:

$$\text{(BMV-2)} \quad V_n(\Delta)V(P, Q, \Delta^{n-2}) \leq V(P, \Delta^{n-1})V(Q, \Delta^{n-1}).$$

Conjecture

Let Δ be an n-dimensional convex body satisfying (BMV-2) for arbitrary convex bodies P, Q. Then Δ is an n-simplex.

Main Results

(BMV-2) $V_n(\Delta)V(P, Q, \Delta^{n-2}) \leq V(P, \Delta^{n-1})V(Q, \Delta^{n-1})$

Theorem

Let Δ be an n-dimensional convex body satisfying (BMV-2) for any bodies P, Q. Then

1. *Δ is indecomposable, i.e. if $\Delta = \Delta_1 + \Delta_2$ then $\Delta_1 \sim \Delta_2$.*
2. *Δ has no strict points, i.e. points not lying on a boundary segment.*
3. *If Δ is a polytope then Δ is an n-simplex.*

Main Results

(BMV-2) $V_n(\Delta)V(P, Q, \Delta^{n-2}) \leq V(P, \Delta^{n-1})V(Q, \Delta^{n-1})$

Theorem

Let Δ be an n-dimensional convex body satisfying (BMV-2) for any bodies P, Q. Then

1. *Δ is indecomposable, i.e. if $\Delta = \Delta_1 + \Delta_2$ then $\Delta_1 \sim \Delta_2$.*
2. *Δ has no strict points, i.e. points not lying on a boundary segment.*
3. *If Δ is a polytope then Δ is an n-simplex.*

Idea for 1. If $P = \Delta_1$, $Q = \Delta_2$ we get the reversed A-F. Then $\Delta_1 \sim \Delta_2$.

Main Results

(BMV-2) $V_n(\Delta)V(P,Q,\Delta^{n-2}) \le V(P,\Delta^{n-1})V(Q,\Delta^{n-1})$

Theorem

Let Δ be an n-dimensional convex body satisfying (BMV-2) for any bodies P, Q. Then

1. *Δ is indecomposable, i.e. if $\Delta = \Delta_1 + \Delta_2$ then $\Delta_1 \sim \Delta_2$.*
2. *Δ has no strict points, i.e. points not lying on a boundary segment.*
3. *If Δ is a polytope then Δ is an n-simplex.*

Idea for 2. If $P = [-\xi, \xi]$, $Q = \Delta \setminus$ "cup" then $V_n(\Delta) > V(Q, \Delta^{n-1})$, but

$$V(P, Q, \Delta^{n-2}) = V(Q|\xi^\perp, (\Delta|\xi^\perp)^{n-2}) = V(P, \Delta^{n-1}).$$

Main Results

(BMV-2) $V_n(\Delta)V(P,Q,\Delta^{n-2}) \le V(P,\Delta^{n-1})V(Q,\Delta^{n-1})$

Theorem

Let Δ be an n-dimensional convex body satisfying (BMV-2) for any bodies P, Q. Then

1. *Δ is indecomposable, i.e. if $\Delta = \Delta_1 + \Delta_2$ then $\Delta_1 \sim \Delta_2$.*
2. *Δ has no strict points, i.e. points not lying on a boundary segment.*
3. *If Δ is a polytope then Δ is an n-simplex.*

Idea for 3. Let P be Δ with a "moving facet". Then (BMV-2) is a variational problem which implies $P \sim \Delta$. Hence, Δ is a cone over a moving facet, for every facet. Thus Δ is a simplex.

Bezout Inequality and Projections

Special case of BMV-2. Let $P=[0,\xi]$, $Q=[0,\eta]$ for $\xi,\eta\in S^{n-1}$, $\xi\cdot\eta=0$. Then

$$\frac{n}{n-1}V_n(\Delta)V_{n-2}(\Delta|(\xi,\eta)^\perp)\le V_{n-1}(\Delta|\xi^\perp)V_{n-1}(\Delta|\eta^\perp)$$

for any n-simplex Δ.

Bezout Inequality and Projections

Special case of BMV-2. Let $P = [0, \xi]$, $Q = [0, \eta]$ for $\xi, \eta \in S^{n-1}$, $\xi \cdot \eta = 0$. Then

$$\frac{n}{n-1} V_n(\Delta) V_{n-2}(\Delta | (\xi, \eta)^\perp) \le V_{n-1}(\Delta | \xi^\perp) V_{n-1}(\Delta | \eta^\perp)$$

for any n-simplex Δ.

Lemma (Giannopoulos, Hartzoulaki, Paouris) For any convex body D

$$\frac{n}{n-1} V_n(D) V_{n-2}(D | (\xi, \eta)^\perp) \le 2 V_{n-1}(D | \xi^\perp) V_{n-1}(D | \eta^\perp).$$

Relaxing the Bezout Inequality

Problem: Find the smallest constant $c_{n,r} > 0$ such that

$$V_n(D)^{r-1} V(P_1, \ldots, P_r, D^{n-r}) \leq c_{n,r} \prod_{i=1}^{r} V(P_i, D^{n-1}).$$

holds for arbitrary bodies $P_1, \ldots, P_r$ and D.

Relaxing the Bezout Inequality

Problem: Find the smallest constant $c_{n,r} > 0$ such that

$$V_n(D)^{r-1} V(P_1, \dots, P_r, D^{n-r}) \le c_{n,r} \prod_{i=1}^{r} V(P_i, D^{n-1}).$$

holds for arbitrary bodies $P_1, \dots, P_r$ and D.

Theorem

- $c_{n,r} = r^{r-1}$ when $P_1, \dots, P_r$ are zonoids,
- $c_{n,r} \le n^{r/2} r^{r-1}$ when $P_1, \dots, P_r$ are symmetric,
- $c_{n,r} \le n^r r^{r-1}$ when $P_1, \dots, P_r$ are arbitrary,
- $c_{2,2} = 2$.

Relaxing the Bezout Inequality

Problem: Find the smallest constant $c_{n,r} > 0$ such that

$$V_n(D)^{r-1} V(P_1, \dots, P_r, D^{n-r}) \leq c_{n,r} \prod_{i=1}^{r} V(P_i, D^{n-1}).$$

holds for arbitrary bodies $P_1, \dots, P_r$ and D.

Theorem

- $c_{n,r} = r^{r-1}$ when $P_1, \dots, P_r$ are zonoids,
- $c_{n,r} \leq n^{r/2} r^{r-1}$ when $P_1, \dots, P_r$ are symmetric,
- $c_{n,r} \leq n^r r^{r-1}$ when $P_1, \dots, P_r$ are arbitrary,
- $c_{2,2} = 2$.

— The End —