Toric Geometry in Coding Theory

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Part I

What is Coding Theory?

Basic Definitions from Coding Theory

Let $\mathbb{K} = \mathbb{F}_q$ be a finite field of q elements (the alphabet).
Linear Code: $\mathcal{C} \subset \mathbb{K}^n$ a subspace, elements of $\mathcal{C}$ are the codewords.

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Parameters:

- n the length of $\mathcal{C}$
- $k = \dim \mathcal{C}$ the dimension of $\mathcal{C}$
- d the minimum distance of $\mathcal{C}$

Hamming distance: $dist(c_1, c_2) = \#$of non-zero entries in $c_1 - c_2$.

$$d = \min_{c \in \mathcal{C} \setminus \{0\}} (\#\text{of non-zero entries in } c)$$

We say that $\mathcal{C}$ is a $[n, k, d]_q$-code.

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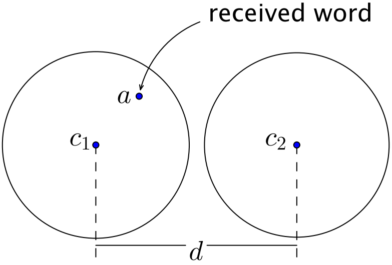

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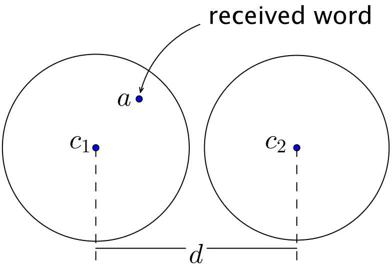

Goal: Given n, k construct an $[n, k, d]_q$-code with largest d.

Evaluation Codes

Let X be an algebraic variety over $\mathbb{K}$. We fix $Z = \{p_1, \ldots, p_n\} \subset X(\mathbb{K})$ and

$\mathcal{L}$ = f. dim. space of rational functions over $\mathbb{K}$, regular on Z.

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Evaluation Map:

$$\mathrm{ev}_Z : \mathcal{L} \to \mathbb{K}^n \quad f \mapsto (f(p_1), \dots, f(p_n)).$$

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Problem: Compute the minimum distance $d(\mathcal{C}_{Z,\mathcal{L}})$.
Equivalently, what is the largest number of points of Z that can lie on the hypersurface $f = 0$ for some $f \in \mathcal{L}$ such that $f_{|Z} \neq 0$?

Classical Examples

Reed-Solomon Codes: Let $X = \mathbb{P}^1$, $Z = \{p_1, \ldots, p_n\} \subset \mathbb{K}$, and

$$\mathcal{L} = \{f \in \mathbb{K}[x] \mid \deg f \leq m\}, \quad \text{where } m < n.$$

Then $\mathcal{C}_{Z,\mathcal{L}}$ is a $[n, m+1, d]_q$-code, where $d = n - m$.

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Goppa Codes: Let X be alg curve, $Z = \{p_1, \dots, p_n\} \subset X(\mathbb{K})$,
$\mathcal{L} = \mathcal{L}(D)$ for a divisor D with $\deg D = m$ and $\operatorname{supp} D \cap Z = \varnothing$.

Assume $2g - 2 < m < n$, where g is the genus of X.
Then $\mathcal{C}_{Z,\mathcal{L}}$ is a $[n, m - g + 1, d]_q$-code, where $d \ge n - m$.

Part II

Toric Codes

Why toric codes?

- Toric codes make connections:

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- Toric codes make connections:
  Alg Geom $\longleftrightarrow$ Coding $\longleftrightarrow$ Lattice Polytopes
- Toric codes are champions:
  (Recent improvements of `www.codetables.de`)

  Toric Codes:

  $[49, 12, 28]_8$ J. Little (2011)
  $[36, 19, 12]_7$ G. Brown, A. Kasprzyk (2012)

  Codes from T-varieties and Generalized Toric Codes:

  $[49, 8, 34]_8$ A. Carbonara, J. Murillo, A. Ortiz (2010)
  $[66, 19, 30]_7$ N. Ilten, H. Süss (2011)
  $[49, 13, 27]_8$, G. Brown and A. Kasprzyk, S. (2013)

  and six more...

Definition of toric codes

Let X be a toric variety over $\bar{\mathbb{K}}$, $\dim X = \ell$.
Let $Z = \mathbb{T} = (\mathbb{K}^*)^\ell$, $\mathcal{L} =$ sections of a line bundle over $\mathbb{K}$ on X.

Explicitly, let P be a lattice polytope in $\mathbb{R}^\ell$, and let $P_{\mathbb{Z}} = P \cap \mathbb{Z}^\ell$.
It defines a finite dimensional space of Laurent polynomials:

$$\mathcal{L}_P = \operatorname{span}_{\mathbb{K}}\{t^a \mid a \in P_{\mathbb{Z}}\}, \text{ where } t^a = t_1^{a_1} \cdots t_\ell^{a_\ell}.$$

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Observe: The evaluation map $ev_\mathbb{T} : \mathcal{L}_P \to \mathbb{K}^n$ is injective iff points in $P_\mathbb{Z}$ are distinct in $(\mathbb{Z}_{q-1})^\ell$.
Then $\mathcal{C}_\mathcal{P}$ has parameters $n = (q-1)^\ell$, $k = |P_\mathbb{Z}|$. What about d?

Explicit Answers

Serre (1989) If $P = m\triangle_\ell$, where $\triangle_\ell$ is the standard simplex then

$$d(\mathcal{C}_P) = (q-1)^{\ell-1}(q-1-m).$$

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S-S (2010) For any polytopes P, Q

$$d(\mathcal{C}_{P\times Q}) = d(\mathcal{C}_P)d(\mathcal{C}_Q).$$

$$d(\mathcal{C}_{mPyr(Q)}) = (q-1)d(\mathcal{C}_{mQ}), \text{ for any } m = 1, 2, 3, \ldots$$

which produces many more explicit answers.

Bounds on $d(\mathcal{C}_{\mathcal{P}})$

Toric Surface Codes
Assume $\ell = 2$, so P is a lattice polygon. We have

Hasse-Weil Bound: If Y is an irreducible curve over $\mathbb{K}$ then

$$q + 1 - 2g\sqrt{q} \leq |Y(\mathbb{K})| \leq q + 1 + 2g\sqrt{q},$$

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where g is the genus of Y.

Little-Schenck used this to show that if $q \gg 0$ then for any non-zero $f, g \in \mathcal{L}_P$

f has more irr factors than $g \iff f$ has more zeroes in $\mathbb{T}$ than g

Newton polytopes and Minkowski Sum

Let f be a Laurent polynomial $f \in \mathbb{K}[t_1, \ldots, t_\ell]$. Let $P(f)$ be its Newton Polytope: $P(f) =$ conv.hull{ exponents of f} $\subset \mathbb{R}^\ell$
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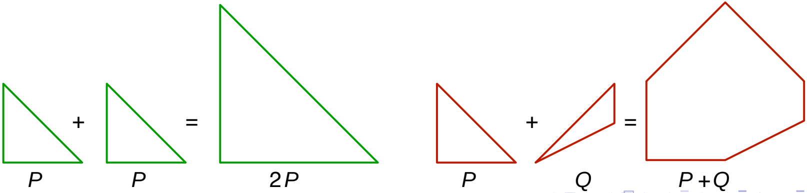
The Minkowski sum of polytopes P, Q in $\mathbb{R}^\ell$ is

$$P + Q = \{p + q \in \mathbb{R}^\ell \mid p \in P,\ q \in Q\}.$$



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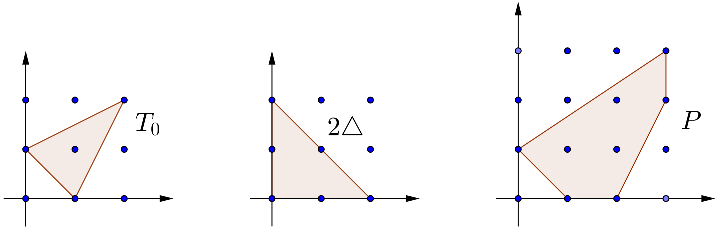

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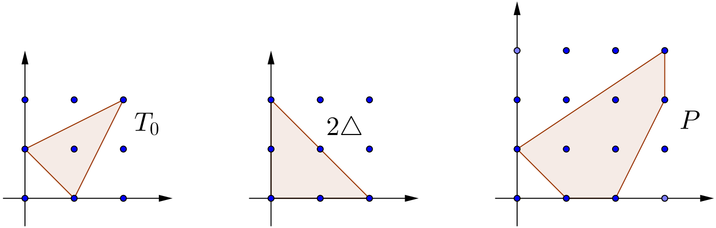

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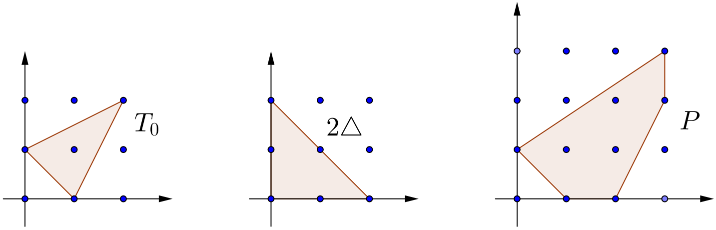

$L(T_0) = 1$ $\quad$ $L(2\triangle) = 2$

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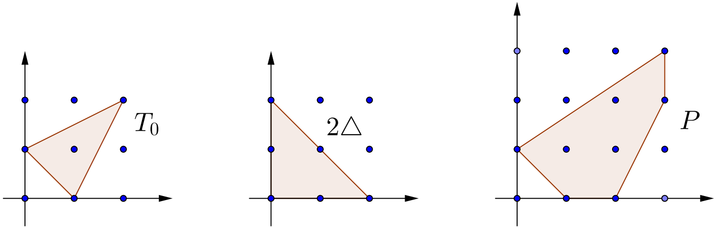

$L(T_0) = 1$ $\quad$ $L(2\triangle) = 2$ $\quad$ $L(P) = 3$

Bounds on $d(\mathcal{C}_{\mathcal{P}})$

Theorem (S-S)

Suppose $\ell = 2$, P a lattice polygon, and $q > \alpha(P)$. If $f \in \mathcal{L}_P$ has largest number of factors then the factors are "simple enough". This implies

$$d(\mathcal{C}_P) \geq (q-1)(q-1-L(P)) - (2\sqrt{q}-1)$$

("Simple enough" means $P(f_i) =$ primitive segment or $\triangle$ or T_0.)
(No $2\sqrt{q}-1$ term if no T_0 appears in a maximal factorization.)

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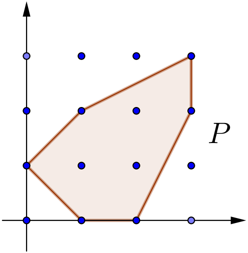

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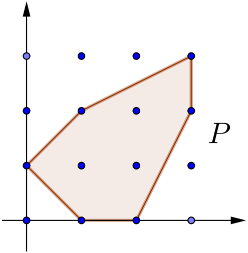

Here $L(P) = 3$. Our bound says

$$d(\mathcal{C}_P) \geq (q-1)(q-1-3) \text{ for } q \geq 37.$$

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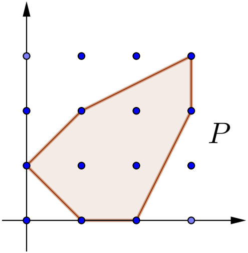

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In fact,

$$d(\mathcal{C}_P) = (q-1)(q-1-3) + 2$$

for $q \geq 5$, $q \neq 8$. The bound is attained at $x(x-a)(y-b)(y-c)$, for $a, b, c \in \mathbb{K}^*$.

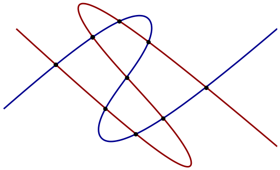
For $q = 8$, $d(\mathcal{C}_P) = (q-1)(q-1-3)$, the bound is attained at $x^2 + y + x^3y^3$.

Part III

Toric Complete Intersection Codes

Chasles' Theorem (1860)

Let C_1, C_2 be two cubics intersecting in nine points $\{p_1, \ldots, p_9\}$. Then any cubic E containing $\{p_1, \ldots, p_8\}$ must contain p_9 as well.



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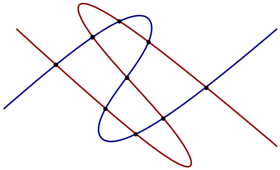

Proof: $C_1 = \{f_1 = 0\}$, $C_2 = \{f_2 = 0\}$ intersect in $S = \{p_1, \ldots, p_9\}$. Let $E = \{h = 0\}$.

Since $\deg h = 3$, the form

$$\omega_h = \frac{h}{f_1 f_2} dt_1 \wedge dt_2 \quad \text{has no poles at infinity.}$$

By Residue Theorem

$$\sum_{i=1}^{9} \operatorname{res}_{p_i} \omega_h = \sum_{i=1}^{9} \frac{h(p_i)}{J_f(p_i)} = 0,$$

so if $h(p_i) = 0$ for $1 \leq i \leq 8$ then $h(p_9) = 0$.

Toric Complete Intersections (TCI)

Fix ℓ full-dim'l lattice polytopes $P_1, \dots, P_\ell$ in $\mathbb{R}^\ell$ and let $P = \sum P_i$. Let $f_1, \dots, f_\ell$ be Laurent polynomials with $P(f_i) = P_i$. We say

$$Z = \{p \in (\bar{\mathbb{K}}^*)^\ell \mid f_1(p) = \cdots = f_\ell(p) = 0\}$$

is a toric compete intersection if $|Z| = V(P_1, \dots, P_\ell)$ (i.e. the BKK bound on the intersection number is attained).

Equivalently, there are ℓ hypersurfaces with Newton polytopes $P_1, \dots, P_\ell$ in the toric variety X_P with transversal intersections Z lying in the dense orbit of X_P.

Toric Residue Theorem

Let $Z = \{p_1, \ldots, p_n\}$ be a TCI given by $f_1 = \cdots = f_\ell = 0$ with polytopes $P_1, \ldots, P_\ell$. Let P° be the interior of $P = \sum P_i$.

Theorem [Khovanskii,1978]: For any $h \in \mathcal{L}(P^\circ)$ we have

$$\sum_{p \in Z} \operatorname{res}_p \left(\frac{h}{f_1 \cdots f_\ell} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_\ell}{t_\ell} \right) = \sum_{p \in Z} \frac{h(p)}{J_f^{\mathbb{T}}(p)} = 0,$$

where $J_f^{\mathbb{T}} = \det(t_j \partial f_i / \partial t_j)$ is the toric Jacobian of the f_i.

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Corollary: If $h \in \mathcal{L}(P^\circ)$ and $h(p_i) = 0$ for $1 \leq i \leq n-1$ then $h(p_n) = 0$.

TCI codes

Let $Z = \{p_1, \ldots, p_n\}$ be a TCI given by $f_1 = \cdots = f_\ell = 0$ with polytopes $P_1, \ldots, P_\ell$. Let P° be the interior of $P = \sum P_i$. Assume $Z \subset \mathbb{T}$.

For any $A \subset P^\circ$ define $\mathcal{L}_A = \operatorname{span}_{\mathbb{K}}\{t^a \mid a \in A_{\mathbb{Z}}\}$.

Evaluation map: $ev_Z : \mathcal{L}_A \to \mathbb{K}^n, \quad f \mapsto (f(p_1), \ldots, f(p_n))$

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Problem: Give bounds on the minimum distance $d(\mathcal{C}_{Z,A})$ in terms of A and the polytope P.

Bound on the minimum distance

Proposition: Let $A = P^\circ$. Then $d(\mathcal{C}_{Z,P^\circ}) \geq 2$.

(since any $0 \neq h \in \mathcal{L}(P^\circ)$ has at most $|Z| - 2$ zeroes in Z by the Residue Theorem)

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Example [Gold–Little–Schenck]: Let $Z \subset \mathbb{P}^\ell$ defined by hypersurfaces of degrees $d_1, \ldots, d_\ell$. Let $\mathcal{L}_A = \mathcal{L}(a)$ and $a + m \leq \sum d_i - (\ell + 1)$. Then $d(\mathcal{C}_{Z,a}) \geq m + 2$.

This bound is sharp!

Improved bound for "generic" TCI

Let Q be a lattice polytope. We say that $Z \subset \mathbb{T}$ is Q-generic if any $T \subset Z$ with $|T| = |Q_{\mathbb{Z}}|$ the evaluation map $\mathrm{ev}_T : \mathcal{L}_Q \to \mathbb{K}^{|T|}$ is an isomorphism.

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Theorem 2 [S]: Let Z be a Q-generic TCI. Suppose $A + mQ \subset P^\circ$ for some $m \geq 0$. Then

$$d(\mathcal{C}_{Z,A}) \geq (|Q_{\mathbb{Z}}| - 1)m + 2.$$

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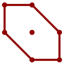
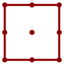
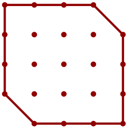
Example [Ballico–Fontanari]: Let $Z \subset \mathbb{P}^\ell$ be $\triangle_\ell$-generic defined by hypersurfaces of degrees $d_1, \ldots, d_\ell$. Let $\mathcal{L}_A = \mathcal{L}(a)$ and $a + m \leq \sum d_i - (\ell + 1)$. Then $d(\mathcal{C}_{Z,a}) \geq \ell m + 2$.

Small example

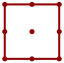
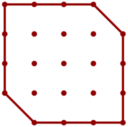
Let $P_1 =$  and $P_2 =$ . We have $P =$ 

Let $\mathbb{K} = \mathbb{F}_{13}$ and consider the system

$$\begin{cases} f_1 = x - y - 6x^2 - 2xy + 6y^2 - 3x^2y + 2xy^2 \\ f_2 = 1 - x - 4y - x^2 + 2xy - 2y^2 - 2x^2y - 3xy^2 + x^2y^2. \end{cases}$$

The system has $8 = V(P_1, P_2)$ solutions in $(\mathbb{F}_{13}^*)^2$.

Small example

Let $P_1 =$  and $P_2 =$ . We have $P =$ 

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The system has $8 = V(P_1, P_2)$ solutions in $(\mathbb{F}_{13}^*)^2$.

Let $A = Q =$ . Then $A + Q \subset P^\circ$, so

$$d(\mathcal{C}_{S,\mathcal{L}(A)}) \geq (4 - 1) + 2 = 5.$$

Also $\dim(\mathcal{C}_{S,\mathcal{L}(A)}) = |A_{\mathbb{Z}}| = 4$, so we get a $[8, 4, 5]$-code over $\mathbb{F}_{13}$.