Evaluation codes and their duals

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Evaluation Codes

Let $\mathbb{K} = \mathbb{F}_q$ finite field and $S = \mathbb{K}[t_1, \ldots, t_s]$ polynomial ring.

Fix (1) $\mathcal{L} \subset S$ subspace; (2) $X \subset \mathbb{K}^s$ finitely many points.

Define the evaluation map

$$\mathrm{ev}_X : \mathcal{L} \to \mathbb{K}^{|X|} \quad f \mapsto (\, f(x) \mid x \in X).$$

Then the evaluation code is $\mathcal{L}_X := \mathrm{ev}_X(\mathcal{L})$.

Note: $\mathrm{Ker}(\mathrm{ev}_X) = \mathcal{L} \cap I$, where $I = I(X)$ is the vanishing ideal of X.

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Parameters of $\mathcal{L}_X$:

- Length $n = |X|$
- Dimension $k = \dim(\mathcal{L}/\mathcal{L} \cap I)$
- minimum distance $d = n - \max\{|V(f) \cap X| : f \in \mathcal{L} \setminus \mathcal{L} \cap I\}$ where $V(f)$ is the set of $\mathbb{F}_q$-zeros of f.

Motivational example: Reed-Solomon Code

Let $S = \mathbb{K}[t]$ and fix $X \subset \mathbb{K}$.

Let $\mathcal{L}(r) \subset S$ polynomials of degree at most r, for some $r < n = |X|$.

Then $\mathcal{L}(r)_X = \mathrm{ev}_X(\mathcal{L}(r))$ is the $[n, r+1, n-r]$ Reed-Solomon code.

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Theorem (Duality for Reed-Solomon Codes)

The dual of the Reed-Solomon code $\mathcal{L}(r)_X$ is monomially equivalent to the Reed-Solomon code $\mathcal{L}(n-r-2)_X$

$$\mathcal{L}(r)_X{}^{\perp} = \lambda \cdot \mathcal{L}(n-r-2)_X, \text{ for some } \lambda \in (\mathbb{K}^*)^n.$$

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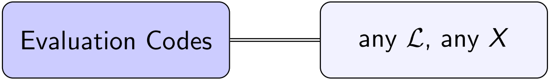
Main Question: How does this phenomenon extend to other (more general) classes of evaluation codes?

We look at duality for the following classes of codes:



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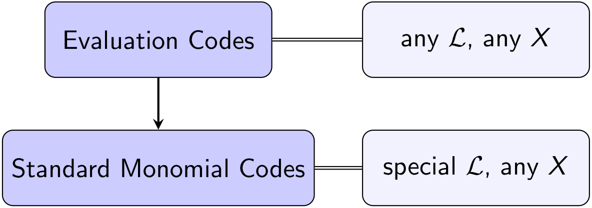

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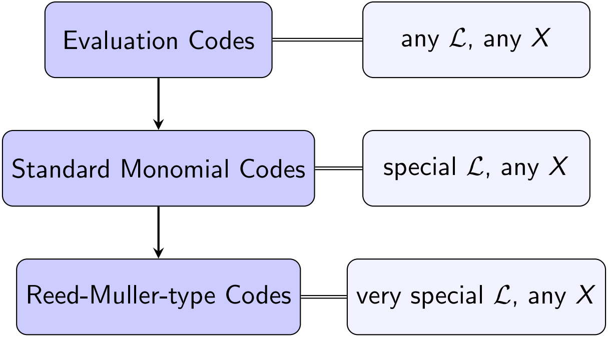

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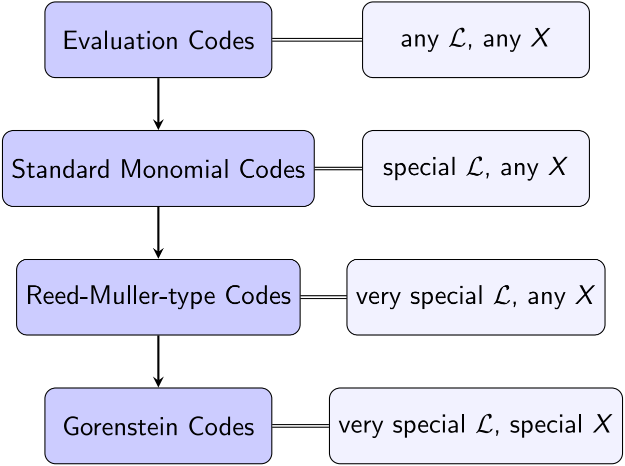

Duality for Evaluation Codes

Let $\mathcal{L} \subset S$ and $X \subset \mathbb{K}^s$ as before.

Question: Given $\mathcal{L} \subset S$, what subspace $\mathcal{L}^\perp \subset S$ can we take so that

$$\mathrm{ev}_X(\mathcal{L}^\perp) = \mathrm{ev}_X(\mathcal{L})^\perp?$$

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$$\mathrm{tr}_X : \mathcal{L} \to \mathbb{K} \quad f \mapsto \sum_{x \in X} f(x).$$

Answer: We can take

$$\mathcal{L}^\perp = \mathrm{Ker}(\mathrm{tr}_X) : \mathcal{L} = \{g \in S \mid fg \in \mathrm{Ker}(\mathrm{tr}_X) \text{ for all } f \in \mathcal{L}\}.$$

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Reason: We have $0 = (\mathrm{ev}(f) \cdot \mathrm{ev}(g)) = \sum_{x \in X} f(x)g(x) = \mathrm{tr}(fg)$.

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This is not explicit enough, though...

Standard Monomials

Fix a graded monomial order $\prec$ on $S = \mathbb{K}[t_1, \ldots, t_s]$ and let $X \subset \mathbb{K}^s$.
Let $I = I(X)$ be the vanishing ideal of X.

Initial ideal $\mathrm{in}_\prec(I) = \langle \mathrm{in}_\prec(f) \mid f \in I \rangle = \langle \mathrm{in}_\prec(g) \mid g \in \mathcal{G} \rangle$

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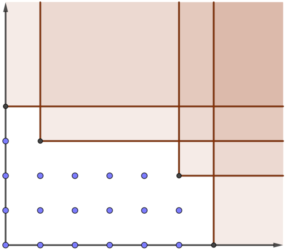

- $|\Delta_\prec| = \dim_\mathbb{K} S/I = |X| = n$
- $\Delta_\prec$ produces a basis for S/I
- $\mathrm{ev}_X : \mathbb{K}\Delta_\prec \xrightarrow{\sim} \mathbb{K}^n$ linear isomorphism
- $\exists$ unique $f_i \in \mathbb{K}\Delta_\prec$ with $\mathrm{ev}_X(f_i) = e_i$, standard indicator functions

Duality for Evaluation Codes

Theorem (López-S-Villarreal '21)

Consider an evaluation code $\mathrm{ev}_X(\mathcal{L})$. Let $\mathcal{L}^\perp = \mathrm{Ker}(\mathrm{tr}_X) : \mathcal{L}$, as before. Then

$$\mathrm{ev}_X(\mathcal{L}^\perp \cap \mathbb{K}\Delta_\prec) = \mathrm{ev}_X(\mathcal{L})^\perp.$$

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Example: If $\mathcal{L} \subset S$ is spanned by monomials and I is a binomial ideal then $\mathcal{L}_X$ is a standard monomial code. This covers Reed-Muller codes, (generalized) toric codes, monomial codes on degenerate tori, etc.

Duality for Standard Monomial Codes

Let $\mathcal{L}_1 = \mathbb{K}M_1$ and $\mathcal{L}_2 = \mathbb{K}M_2$ for $M_1, M_2 \subset \Delta_{\prec}$ and $\mathcal{C}_1 = \mathrm{ev}_X(\mathcal{L}_1)$ and $\mathcal{C}_2 = \mathrm{ev}_X(\mathcal{L}_2)$ the corresponding standard monomial codes.

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Theorem (López-S-Villarreal '21)

Let t^e be the largest monomial in $\Delta_\prec$. Assume t^e appears in each standard indicator function f_i. Then if M_1, M_2 satisfy

- $|M_1| + |M_2| = |X|$
- $t^e \notin M_1 M_2$ *(set of pairwise products)*

then

$$\mathcal{C}_1^\perp = \lambda \cdot \mathcal{C}_2,$$

where $\lambda = (\mathrm{lc}(f_1), \ldots, \mathrm{lc}(f_n)) \in (\mathbb{K}^*)^n$.

Duality for Standard Monomial Codes

Example: Let $\mathbb{K} = \mathbb{F}_7$, $X = \{(1, \pm 1), (2, \pm 1), (4, \pm 1)\} \subset \mathbb{K}^2$.
Then $I = \langle t_1^3 - 1, t_2^2 - 1 \rangle$ and $\text{in}_\prec(I) = \langle t_1^3, t_2^2 \rangle$.

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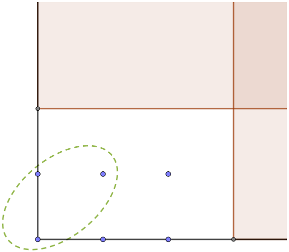

Here $\Delta_\prec = \{1, t_1, t_2, t_1^2, t_1 t_2, t_1^2 t_2\}$.

Let $M = \{1, t_2, t_1 t_2\}$ and $\mathcal{C} = \mathrm{ev}_X(\mathbb{K}M)$.

Note:
- $|M| + |M| = |X|$
- $t_1^2 t_2 \notin MM$

Thus, $\mathcal{C}^\perp = \lambda \cdot \mathcal{C}$, where $\lambda = (1, 2, -3, -1, -2, 3) = \mathrm{ev}_X(t_1 t_2)$.
This is a λ-self-dual code.

Reed-Muller-type Codes

As before, $I \subset S$ is the vanishing ideal of X and $\Delta_{\prec}$ the set of standard monomials relative to a graded monomial order $\prec$.

Let $S_{\leq r}$ polynomials of total degree at most r.

- affine Hilbert function: $H_I(r) = \dim_{\mathbb{K}}(S_{\leq r}/I_{\leq r})$
- regularity: r_0 smallest such that $H_I(r) = |X|$ for $r \geq r_0$
- local v-numbers: $v_{\mathfrak{p}}(I) = \min\{\deg f \mid (I : f) = \mathfrak{p}\}$, for $\mathfrak{p} \in \mathrm{Ass}(I)$

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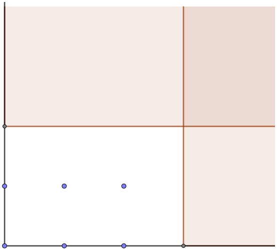

- $H_I(r) = H_{\mathrm{in}_{\prec}(I)}(r) = |\Delta_{\prec} \cap S_{\leq r}|$
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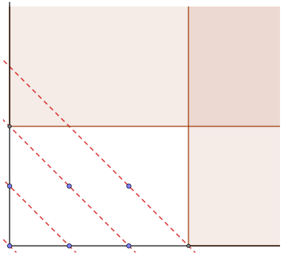

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Duality for Reed-Muller-type Codes

Let I be the vanishing ideal of $X \subset \mathbb{K}^s$ and with affine Hilbert function $H_I(r)$ and regularity r_0.

Definition: Reed-Muller-type code $\mathcal{C}_X(r) = \text{ev}_X(S_{\leq r})$, for $-1 \leq r \leq r_0$

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Theorem (López-S-Villarreal '21) *The following are equivalent:*

(a) $H_I(r) + H_I(r_0 - r - 1) = |X|$ *for all* $-1 \leq r \leq r_0$ *and* $r_0 = v_{\mathfrak{p}}(I)$ *for all* $\mathfrak{p} \in Ass(I)$

(b) *The dual* $\mathcal{C}_X(r)^\perp$ *is monomially equivalent to* $\mathcal{C}_X(r_0 - r - 1)$ *for all* $-1 \leq r \leq r_0$

In this case,

$$\mathcal{C}_X(r)^\perp = \lambda \cdot \mathcal{C}_X(r_0 - r - 1),$$

where $\lambda = (\mathrm{lc}(f_1), \ldots, \mathrm{lc}(f_n)) \in (\mathbb{K}^*)^n$.

Note: $H_I(r) = \dim \mathcal{C}_X(r)$, so the "symmetry" of H_I is a necessary condition for duality.

Gorenstein Rings

For $X \subset \mathbb{K}^s$ let $\overline{X} = \{(1 : x) \in \mathbb{P}^s \mid x \in X\}$ corresp. points in $\mathbb{P}^s$.

If $I = I(X) \subset S = \mathbb{K}[t_1, \ldots, t_s]$ is the vanishing ideal of X then
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Recall: Let $A = \bigoplus_{i \geq 0}^{k} A_i$ be Artinian graded $\mathbb{K}$-algebra, $A_+ = \bigoplus_{i>0} N_i$, and $\mathrm{Soc}(A) = (0 :_A A_+)$ the socle of A.

We say A is Gorenstein if $\mathrm{Soc}(A) \simeq \mathbb{K}$. In this case $\mathrm{Soc}(A) = A_k$ and the multiplication $A_i \times A_{k-i} \to A_k$ is a perfect pairing.

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Definition: We say X is Gorenstein if $\overline{S}/\langle \overline{I}, t_0 \rangle$ is Gorenstein.

Example: If X is a complete intersection then it is Gorenstein.

Gorenstein Codes

Let $\mathcal{C}_X(r) = \mathrm{ev}_X(S_{\leq r})$ be a Reed-Muller-type code for some $-1 \leq r \leq r_0$, where r_0 is the regularity of $I = I(X)$.

Theorem (López-S-Villarreal '21) *If $X \subset \mathbb{K}^s$ is Gorenstein then $\mathcal{C}_X(r)^\perp$ is monomially equivalent to $\mathcal{C}_X(r_0 - r - 1)$.*

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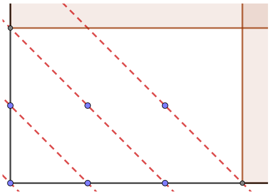
Reason: Follows from the previous criterion since



- homogenization of each f_i mod $(\bar{I}, t_0)$ spans $\mathrm{Soc}(\overline{S}/\langle \bar{I}, t_0\rangle)$;
  also the socle is generated in degree r_0, so $r_0 = \deg f_i$ for all i.
- $\overline{S}/\langle \bar{I}, t_0\rangle$ is Gorenstein
  $\Rightarrow \dim(\overline{S}/\langle \bar{I}, t_0\rangle)_r = \dim(\overline{S}/\langle \bar{I}, t_0\rangle)_{r_0 - r}$
  $\Leftrightarrow |\Delta_\prec \cap S_r| = |\Delta_\prec \cap S_{r_0 - r}|$
  $\Leftrightarrow H_I(r) + H_I(r_0 - r - 1) = |X|$

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Corollary Assume $\operatorname{char} \mathbb{K} = 2$, X is Gorenstein and r_0 is odd.
Then $\mathcal{C}_X(\frac{r_0-1}{2})$ is monomially equivalent to a self-dual code.

Reference

Hiram H. López, Ivan Soprunov, Rafael H.Villarreal *The dual of an evaluation code*. Des. Codes Cryptogr. 89 (2021), no. 7, 1367–1403.

Thank you!