Collections of lattice polytopes with a given mixed volume

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Sparse Polynomial Systems and BKK theorem

Sparse Polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

$$f = \sum_{a \in \mathcal{A}} c_a x^a, \text{ where } x^a = x_1^{a_1} \cdots x_n^{a_n}, \quad c_a \in \mathbb{C}^*.$$

The set of exponents $\mathcal{A} \subset \mathbb{Z}^n$ is the support of f. The convex hull of the support $P = \operatorname{conv}(\mathcal{A})$ is the Newton Polytope of f.

Theorem (Bernstein–Khovanskii–Kushnirenko 1976)

Let $f_1 = \cdots = f_n = 0$ be a generic sparse system with Newton polytopes $P_1, \ldots, P_n$. Then it has exactly $V(P_1, \ldots, P_n)$ isolated solutions in $(\mathbb{C}^*)^n$.

Here $V(P_1, \ldots, P_n)$ is the (lattice) mixed volume of the polytopes $P_1, \ldots, P_n$.

Esterov's Question

Question: Given $m \in \mathbb{N}$ can one describe all n-tuples of lattice polytopes $(P_1, \ldots, P_n)$ such that a generic sparse system $f_1 = \cdots = f_n = 0$ with Newton polytopes $P_1, \ldots, P_n$ has exactly m solutions in $(\mathbb{C}^*)^n$?

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State of the art:

- (Esterov–Gusev '15) $m = 1$ and any $n \geq 1$
- (Esterov–Gusev '16) $m \leq 4$ and $n = 2$
- (Esterov–Gusev '16) $m \leq 4$, any $n \geq 1$, unmixed and spanning
- (Hibi–Tsuchiya '19) $m \leq 4$, any $n \geq 1$, unmixed
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(Esterov, '19) The problem of describing all n-variate generic sparse systems that are solvable in radicals reduces to describing all k-variate generic sparse systems with up to 4 solutions, for $k \leq n$.

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Idea: There exists $b(n, m)$, where $m = V(P_1, \ldots, P_n)$ such that

$$\mathrm{Vol}(P_1 + \cdots + P_n) \leq b(n, m).$$

Now the statement follows from the Lagarias–Ziegler theorem.

Question: How big can the bound $b(n, m)$ be?

Esterov-Gusev '18: $b(n, m) = n^n m^{2^n}$ using Aleksandrov-Fenchel ineq's.

Main Question

Let $K_1, \ldots, K_n$ be convex bodies in $\mathbb{R}^n$ of volume at least 1.

Question: What is the maximum of $\mathrm{Vol}(K_1 + \cdots + K_n)$ when $m = V(K_1, \ldots, K_n)$ is fixed?

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Conjecture: The maximum equals $(m + n - 1)^n$ and is attained when $K_1 = mK$ and $K_2 = \cdots = K_n = K$ with $\text{Vol}(K) = 1$.

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Result: The conjecture is true for $n = 2, 3$. Moreover,

$$\mathrm{Vol}(K_1 + \cdots + K_n) \leq O(m^d)$$

Definition of Mixed Volume

Minkowski addition

$A + B = \{a + b \in \mathbb{R}^n \mid a \in A, b \in B\}$ for any $A, B \subset \mathbb{R}^n$.

Consider compact convex sets $K_1, \ldots, K_n$ in $\mathbb{R}^n$.

The mixed volume $V(K_1, \ldots, K_n)$ is the unique symmetric and multilinear w.r.t. Minkowski addition function satisfying

$$V(K, \ldots, K) = \mathrm{Vol}(K),$$

for any compact convex set $K \subset \mathbb{R}^n$. Here $\mathrm{Vol}(K) = n!\,\mathrm{vol}_n(K)$ a normalization of the Euclidean volume in $\mathbb{R}^n$.

Estimating $\mathrm{Vol}(A+B)$ in term of $m = V(A,B)$

$n = 2$:

Minkowski inequality: $\mathrm{Vol}(A)\,\mathrm{Vol}(B) \le V(A,B)^2$

$$\begin{aligned}\mathrm{Vol}(A+B) = V(A+B, A+B) &= V(A,A) + 2V(A,B) + V(B,B)\\ &= \mathrm{Vol}(A) + 2m + \mathrm{Vol}(B).\end{aligned}$$

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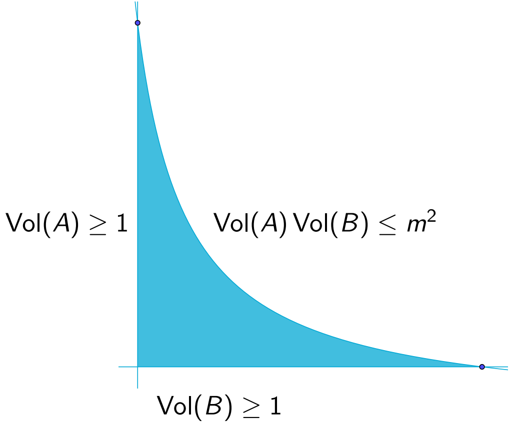

Maximum is attained when $A = mB$, $\mathrm{Vol}(B) = 1$, so

$\mathrm{Vol}(A) = m^2$ and

$\mathrm{Vol}(A+B) = (m+1)^2$

Estimating $\mathrm{Vol}(A+B+C)$ in term of $V(A,B,C)$

$n = 3$:
$$\begin{aligned}\mathrm{Vol}(A+B+C) = {} & V(A,A,A) \\ & +3V(A,A,B)+3V(A,A,C) \\ & +3V(A,B,B)+6V(A,B,C)+3V(A,C,C) \\ & +V(B,B,B)+3V(B,B,C)+3V(B,C,C)+V(C,C,C).\end{aligned}$$

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$n=3$:
$$\begin{aligned}\mathrm{Vol}(A+B+C) &= V(3,0,0)\\ &+3V(2,1,0)+3V(2,0,1)\\ &+3V(1,2,0)+6V(1,1,1)+3V(1,0,2)\\ &+V(0,3,0)+3V(0,2,1)+3V(0,1,2)+V(0,0,3).\end{aligned}$$

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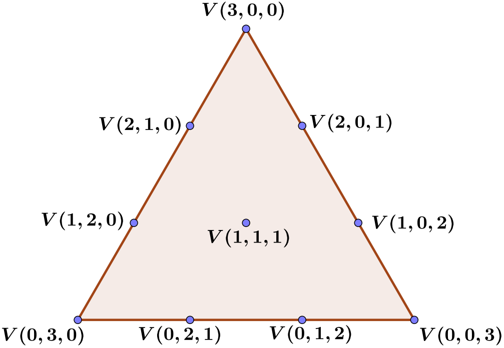


Estimating $\mathrm{Vol}(K_1 + \cdots + K_n)$ in term of $V(K_1, \ldots, K_n)$

In general, for tuple $K = (K_1, \ldots, K_n)$ of convex bodies in $\mathbb{R}^n$, let

$$V_K(p) = V(\underbrace{K_1, \ldots, K_1}_{p_1}, \ldots, \underbrace{K_n, \ldots, K_n}_{p_n})$$

and $\Delta_n = \{p = (p_1, \ldots, p_n) \mid p_i \in \mathbb{Z}_{\geq 0},\ p_1 + \cdots + p_n = n\}$. Then

$$\mathrm{Vol}(K_1 + \cdots + K_n) = \sum_{p \in \Delta_n} \binom{n}{p} V_K(p).$$

We need to maximize this linear function on the mixed volume configuration space:

$$\mathcal{MV}_n = \{(V_K(p))_{p \in \Delta_n} \mid K = (K_1, \ldots, K_n) \text{ with } \mathrm{Vol}(K_i) \geq 1\}.$$

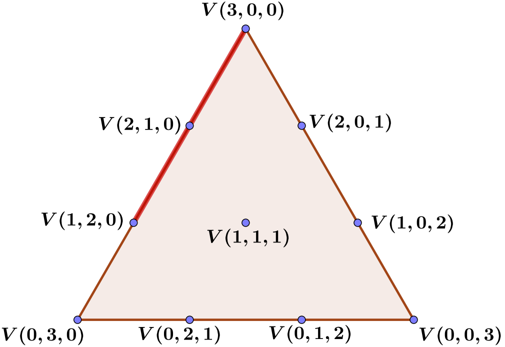
Approximating $\mathcal{MV}_n$ using Aleksandrov-Fenchel relations

Aleksandrov-Fenchel inequality:

$$V(A, A, K_3, \ldots, K_n)V(B, B, K_3, \ldots, K_n) \leq V(A, B, K_3, \ldots, K_n)^2$$

These are log-concavity relations on V_K along standard directions $e_i - e_j$:

$$V_K(p + e_i - e_j)V_K(p + e_j - e_i) \leq V_K(p)^2$$



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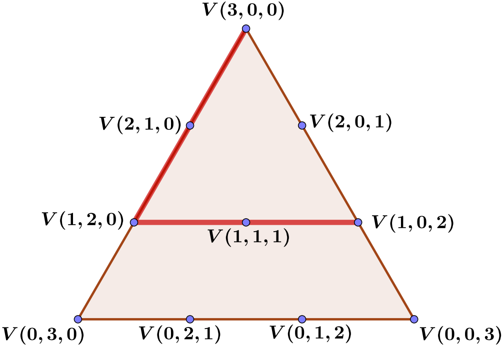

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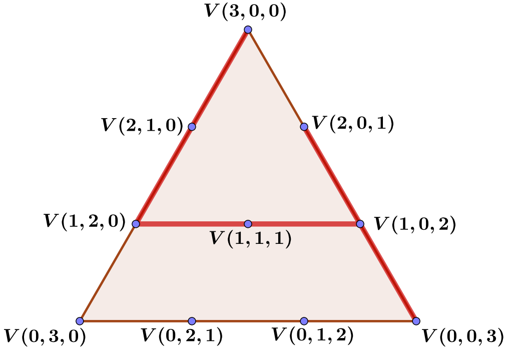

Approximating $\mathcal{MV}_n$ using Aleksandrov-Fenchel relations

We have

$$\mathcal{MV}_n \subset \mathcal{AF}_n := \{(V_p)_{p\in\Delta_n} \mid V_{p+e_i-e_j} V_{p+e_j-e_i} \le V_p^2,\ V_p \ge 1\}.$$

We can turn this into a linear optimization problem by taking log base m

$$\log \mathcal{MV}_n \subset \log \mathcal{AF}_n := \{(v_p)_{p\in\Delta_n} \mid v_{p+e_i-e_j} + v_{p+e_j-e_i} \le 2v_p,\ v_p \ge 0\}.$$

Then we can maximize the convex function in $(v_p,\ p \in \Delta_n)$

$$F := \sum_{p\in\Delta_n} \binom{n}{p} m^{v_p}$$

on the Aleksandrov-Fenchel Polytope $\mathrm{AFP}_n = \log \mathcal{AF}_n \cap \{v_{(1,\dots,1)} = 1\}$.

Approximating $\mathcal{MV}_n$ using Aleksandrov-Fenchel relations

Theorem: ($n = 3$) The maximum of $\mathrm{Vol}(K_1 + K_2 + K_3)$ equals $(m + 2)^3$ where $m = V(K_1, K_2, K_3)$ and is attained when $K_1 = mK_2 = mK_3$ and $\mathrm{Vol}(K_3) = 1$.

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Theorem: The Aleksandrov-Fenchel relations imply the following sharp bound

$$V_p \leq m^{|p|},$$

where $|p| = \prod_{p_i > 0} p_i$ and $m = V_{(1,\ldots,1)}$.

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Corollary: The Aleksandrov-Fenchel relations cannot produce better bound than

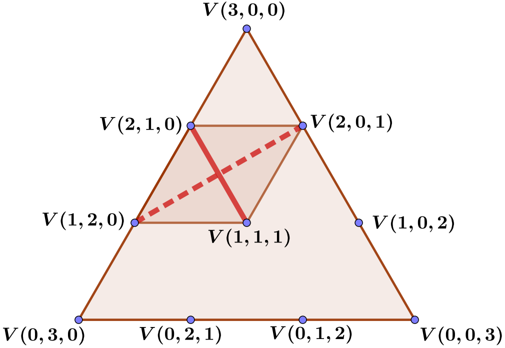
$$V(K_1 + \cdots + K_n) \leq \mathcal{O}(m^{\alpha(n)}),$$

where $3^{(n-2)/3} \leq \alpha(n) \leq 3^{n/3}$.

Approximating $\mathcal{MV}_n$ using Square relations

Square Inequality (Brazitikos, Giannopoulos, Liakopoulos '18)

$V_K(p)V(p+a+b) \le 2V(p+a)V(p+b)$, where $a = e_i - e_\ell, b = e_j - e_\ell$.



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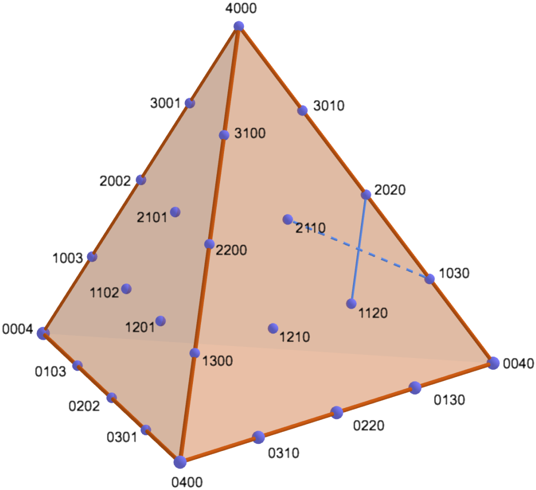

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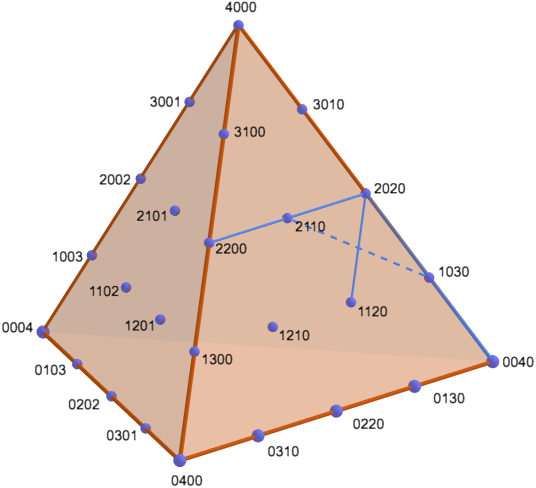

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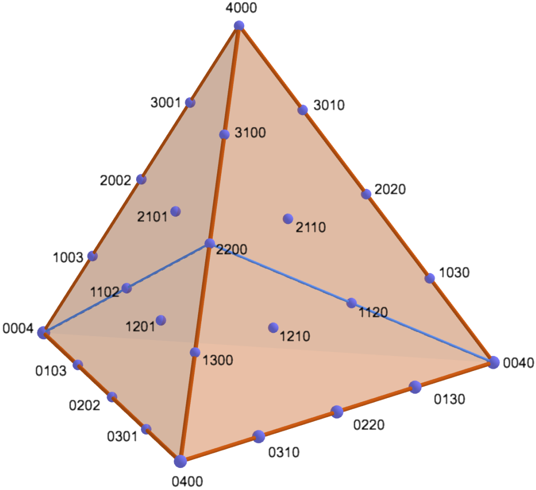

Approximating $\mathcal{MV}_n$ using Square relations

Theorem: The Square relations imply the following bound

$$V_p \leq C(n)\, m^{\max(p)},$$

where $\max(p) = \max_i(p_i)$ and $m = V_{(1,\dots,1)}$. Consequently,

$$\mathrm{Vol}(K_1 + \cdots + K_n) \leq \mathcal{O}(m^n).$$

Further work

Question: Is there a "structural" result for lattice polytopes across all n?

For example,

- (Hofscheier–Katthän–Nill '19) There are only finitely many spanning lattice polytopes of given volume up to lattice equivalence and unit pyramid construction.

- (Balletti–Borger'19) All tuples of n-dim'l lattice polytopes $(P_1, \ldots, P_n)$ with $V(P_1, \ldots, P_n) = |(P_1 + \cdots + P_n)^\circ \cap \mathbb{Z}^n| + 1$ are lattice projections onto $(\Delta_{n-1}, \ldots, \Delta_{n-1})$, except for finitely many exceptions.

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Thank you!

Output: number of trivariate systems with m solutions

Unmixed system = all Newton polytopes are the same
Full-dim'l system = all Newton polytopes are 3-dimensional
Maximal system = no Newton polytope can be increased without changing m

m	# of full-dim'l		# of maximal	running time
	unmixed			
1	1	1	1	
2	3	4	7	$\sim$ 2 hours
3	6	10	21	$\sim$ 1 day
4	17	30	92	$\sim$ 3 days

SageMath code and pictures of Newton polytopes are here:
github.com/christopherborger/mixed_volume_classification