Plücker-type inequalities for mixed areas and intersection numbers

Tulane Math Colloquium

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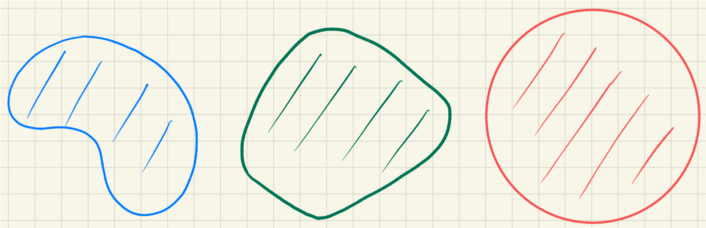
The Volume Polynomial

Isoperimetric Inequality

Isoperimetric problem What plane figure has the largest area when the perimeter is fixed?



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For a disk of radius r we get equality

$$\pi \, \pi r^2 = \left(\frac{2\pi r}{2}\right)^2$$

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This is an instance of a coefficient inequality
for the area polynomial...

The volume polynomial

A convex body is a non-empty compact convex set.

Univariate: Given a convex body $K \subset \mathbb{R}^d$, the function

$$\mathrm{vol}_d(tK) = t^d \mathrm{vol}_d(K)$$

is a homogeneous polynomial of degree d in t.

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Multivariate: (Minkowski 1903)

Given convex bodies $K_1, \dots, K_n \subset \mathbb{R}^d$, the function

$$\mathrm{vol}_d(t_1 K_1 + \cdots + t_n K_n)$$

is a homogeneous polynomial of degree d in $t_1, \dots, t_n$.

Here $t_1 K_1 + \cdots + t_n K_n$ is the Minkowski sum of $t_i K_i$

Minkowski sum

For $A, B \subset \mathbb{R}^d$ define

$$A + B = \{a + b \mid a \in A, b \in B\}$$

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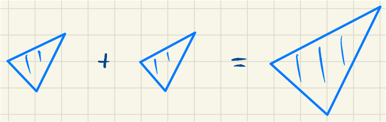

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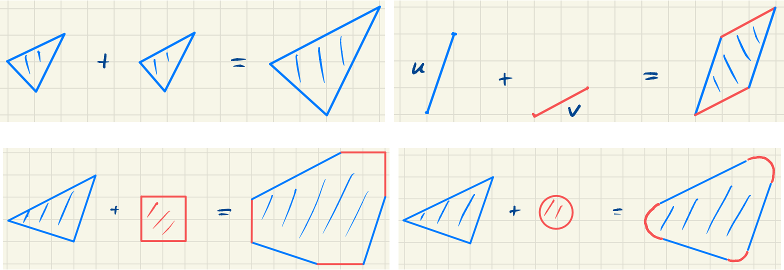

The volume polynomial: Two bodies in $\mathbb{R}^2$

The volume polynomial of $A, B \subset \mathbb{R}^2$ is a quadratic form

$$\mathrm{vol}_2(t_1 A + t_2 B) = \lambda_{11} t_1^2 + 2\lambda_{12} t_1 t_2 + \lambda_{22} t_2^2$$

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We have $\lambda_{11} = \mathrm{vol}_2(A)$, $\lambda_{22} = \mathrm{vol}_2(B)$, and

$$\lambda_{12} = \frac{1}{2}(\mathrm{vol}_2(A+B) - \mathrm{vol}_2(A) - \mathrm{vol}_2(B)) =: \mathsf{V}(A, B) \leftarrow \text{mixed area}$$

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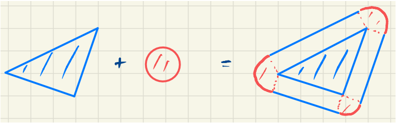
Examples: ▷ $\mathsf{V}(A, A) = \mathrm{vol}_2(A)$

▷ $\mathsf{V}(u, v) = \frac{1}{2}|\det(u, v)|$

▷ $\mathsf{V}(A, \bigcirc) = \frac{1}{2}\mathrm{perim}(A)$



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Note: ▷ When $B = \bigcirc$ this is the isoperimetric inequality!

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$$\mathrm{vol}_2(A)\,\pi \le \left(\frac{1}{2}\mathrm{perim}(A)\right)^2$$

▷ In matrix form: $\mathrm{vol}_2(t_1 A + t_2 B) = t\,\Lambda\,t^T$, for $t = (t_1\ \ t_2)$

Minkowski inequality

$$\det \Lambda = \det \begin{pmatrix} V(A,A) & V(A,B) \\ V(A,B) & V(B,B) \end{pmatrix} \le 0$$

Heine-Shephard Problem

Each n-tuple of bodies $K = (K_1, \ldots, K_n)$ in $\mathbb{R}^d$ defines a volume polynomial

$$\mathrm{vol}_d(t_1 K_1 + \cdots + t_n K_n) = \sum_{1 \le i_1, \ldots, i_d \le n} \mathsf{V}(K_{i_1}, \ldots, K_{i_d}) t_{i_1} \cdots t_{i_d}$$

Its coefficients are the mixed volumes $\mathsf{V}(K_{i_1}, \ldots, K_{i_d})$.

Problem Given n and d, describe the space of all volume polynomials $\mathcal{V}(n, d)$ in terms of coefficient inequalities.

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Example:

$$\mathcal{V}(2,2) = \{t\,\Lambda\,t^T \mid \Lambda \in \mathrm{Sym}_2(\mathbb{R}_{\ge 0}),\ \underbrace{\det \Lambda \le 0}_{\text{Mink ineq}}\}$$

$$\cong \{(x, y, z) \in \mathbb{R}^3_{\ge 0} \mid xz \le y^2\}.$$

Heine-Shephard Problem: Two results

Theorem (Heine 1938) For three convex bodies in $\mathbb{R}^2$ we have

$$\mathcal{V}(3,2) = \Big\{\, t\Lambda t^T \mid \underbrace{\det\Lambda \geq 0}_{\text{det ineq}},\ \underbrace{\det\Lambda_{\hat{i}} \leq 0}_{\text{Mink ineq}},\ i = 1,2,3 \Big\}$$

where $\Lambda \in \mathrm{Sym}_3(\mathbb{R}_{\geq 0})$ and $\Lambda_{\hat{i}}$ are the 2×2 principal minors.

Note: $\mathcal{V}(3,2) \subset \mathbb{R}^6_{\geq 0}$ given by 1 cubic and 3 quadratic inequalities.

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Theorem (Shephard 1960) For two convex bodies in $\mathbb{R}^d$ we have

$$\mathcal{V}(2,d) = \{ c_d t_1^d + \cdots + c_0 t_2^d \mid c_{i-1}c_{i+1} \leq c_i^2 \text{ for } i = 1 \ldots d-1 \}.$$

In other words, it's a space of log-concave sequences of length $d+1$.

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The Heine-Shephard Problem is open in all other cases...

Plücker-type Inequalities

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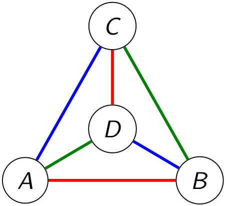

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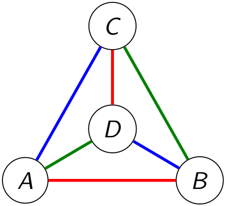

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Theorem (Averkov-S'22) The six mixed volumes satisfy the quadratic inequality

$$\mathrm{V}(A, B)\,\mathrm{V}(C, D) \leq \mathrm{V}(A, C)\,\mathrm{V}(B, D) + \mathrm{V}(A, D)\,\mathrm{V}(B, C)$$

Note: This is the only inequality we know for mixed volumes without repeated bodies. Previously known general inequalities (determinantal, Aleksandrov-Fenchel, Bezout-type, etc.) involve repeated bodies.

Plücker-type inequalities

Why do we call them Plücker-type?

Example: Suppose the $K_i = [0, v_i]$ are segments in a half-plane ordered counterclockwise, $1 \leq i \leq 4$. Consider the matrix

$$M = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix}$$

Then $2\,\mathrm{V}(K_i, K_j) = \det(v_i, v_j) =: v_{ij}$, the maximal minors of M, which satisfy the Grassman-Plücker relation

$$v_{12}v_{34} - v_{13}v_{24} + v_{14}v_{23} = 0$$

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Note: This relation is "linear" in each segment K_i. This implies the theorem for centrally-symmetric bodies, since they are Minkowski sums of segments.

Square-free part of volume polynomials

Recall:

$\mathcal{V}(n,d) :=$ the space of volume polynomials of n bodies in $\mathbb{R}^d$.

Define:

$\mathcal{PV}(n,d) :=$ the space of square-free parts of elements of $\mathcal{V}(n,d)$.

Example: The elements of $\mathcal{V}(n,2)$ are quadratic forms

$$\sum_{i=1}^n \mathrm{V}(K_i,K_i)t_i^2 + 2\sum_{i<j} \mathrm{V}(K_i,K_j)t_it_j$$

whereas the elements of $\mathcal{PV}(n,2)$ are square-free quadratic forms

$$2\sum_{i<j} \mathrm{V}(K_i,K_j)t_it_j$$

Note: $\mathcal{V}(n,d) \to \mathcal{PV}(n,d)$ is a coordinate projection, so information about $\mathcal{PV}(n,d)$ may give some insight about $\mathcal{V}(n,d)$

Inequality description of $\mathcal{PV}(n, d)$

Proposition (Averkov-S'22) ▷ $\mathcal{PV}(d, d) = \mathbb{R}_{\geq 0}$
▷ $\mathcal{PV}(d+1, d) = \mathbb{R}^{d+1}_{\geq 0}$

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Theorem (Averkov-S'22) The Plücker-type inequalities completely describe $\mathcal{PV}(4, 2)$

$$\mathcal{PV}(4,2) = \left\{ 2\sum_{i<j} c_{ij} t_i t_j \;\middle|\; \begin{array}{c} c_{12}c_{34} + c_{13}c_{24} - c_{14}c_{23} \geq 0 \\ c_{12}c_{34} - c_{13}c_{24} + c_{14}c_{23} \geq 0 \\ -c_{12}c_{34} + c_{13}c_{24} + c_{14}c_{23} \geq 0 \end{array} \right\}$$

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Theorem (Averkov-S'22) For any $n \geq 4$ we have

$$\mathcal{PV}(n,2) \subseteq \left\{ 2\sum_{i<j} c_{ij} t_i t_j \mid c_{ij}c_{kl} \leq c_{ik}c_{jl} + c_{il}c_{jk} \text{ for } \{i,j\} \sqcup \{k,l\} \subseteq [n] \right\}$$

Moreover, this containment is proper for $n \geq 8$.

Geometry of $\mathcal{PV}(n, 2)$

Dimension

Theorem (Averkov-S'22) Both $\mathcal{V}(n, 2)$ and $\mathcal{PV}(n, 2)$ have non-empty (relative) interior for $n \geq 2$.

Corollary There are no non-trivial polynomial equations on the coefficients of the area polynomial.

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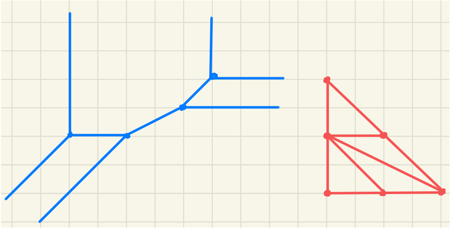
Corollary There are no non-trivial polynomial equations on the coefficients of the area polynomial.

Semi-algebraicity

Theorem (Averkov-S'22) The closure $\overline{\mathcal{PV}(n, 2)} \subset \mathbb{R}^{\binom{n}{2}}$ is a semi-algebraic set, i.e. a set which can be described by a boolean combination of polynomial inequalities.

Applications and Other Directions



Toric and Tropical curves

Fix a finite subset $A \subset \mathbb{Z}^2$, called the support.

Toric:

Laurent polynomial

$$f = \sum_{(a_1,a_2)\in A} \lambda_{a_1,a_2} x^{a_1} y^{a_2}$$

where $\lambda_{a_1,a_2} \in \mathbb{C}^*$

Tropical:

Tropical polynomial

$$f = \min_{(a_1,a_2)\in A} \{\lambda_{a_1,a_2} + a_1 x + a_2 y\}$$

where $\lambda_{a_1,a_2} \in \mathbb{R}$

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Toric curve
$C_f = \{(x,y) \in (\mathbb{C}^*)^2 \mid f(x,y) = 0\}$

Tropical:

Tropical polynomial

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where $\lambda_{a_1,a_2} \in \mathbb{R}$

Tropical curve
$C_f = \{(x,y) \in \mathbb{R}^2 \mid f \text{ not diff at } (x,y)\}$
(It's a 1-dim polyhedral complex in $\mathbb{R}^2$)

Intersection numbers of curve arrangements

Consider n toric/tropical curves intersecting transversely.

Question Are there (algebraic) relations between the $\binom{n}{2}$ pairwise intersection numbers?

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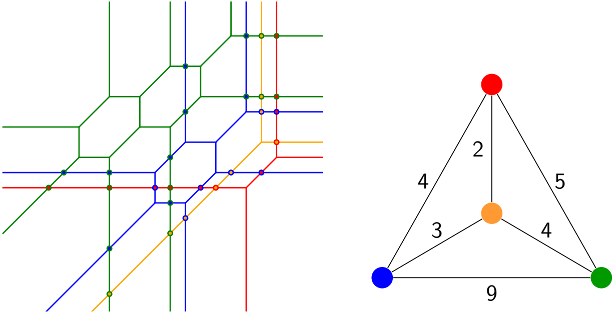

Four tropical curves with six intersection numbers

Intersection numbers are mixed volumes

Let C_1, C_2 be two generic toric/tropical curves with supports A_1 and A_2 and Newton polytopes $P_1 = \text{conv}(A_1)$ and $P_2 = \text{conv}(A_2)$.

Let $I(C_1, C_2) =$ number of intersection points counting multiplicities.

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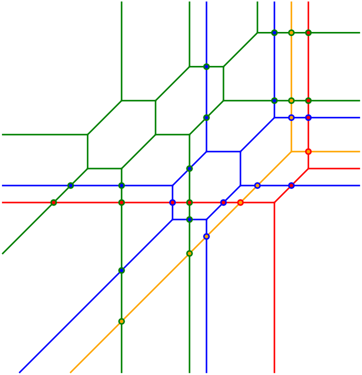
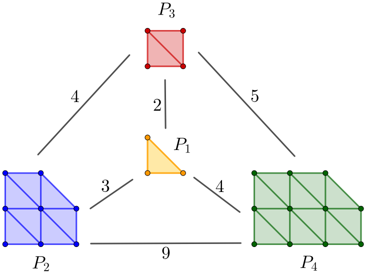
BKK Theorem in dimension 2 (Bernstein–Khovanskii-Kushnirenko'75)

$$I(C_1, C_2) = 2\,\mathrm{V}(P_1, P_2)$$

$\uparrow$ intersection number $\qquad$ $\uparrow$ mixed area





The six intersection numbers $I_C = (2, 3, 4, 4, 5, 9)$ satisfy the Plücker-type inequalities, i.e. the three products

$$2 \cdot 9 = 18, \; 3 \cdot 5 = 15, \; 4 \cdot 4 = 16.$$

satisfy the triangle inequalities.

Intersection numbers of curve arrangements

Each n-tuple C of tropical curves with pairwise transversal intersections defines a vector I_C of $\binom{n}{2}$ intersection numbers.

Define: $\mathcal{I}(n,2) = \{I_C \mid C \text{ transversal arrangement of } n \text{ tropical curves}\}$

By the BBK theorem we have $\mathcal{I}(n,2) \subseteq \mathcal{PV}(n,2) \cap \mathbb{Z}^{\binom{n}{2}}$.

We don't know if these sets are equal. Here is what we know:

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Let $\mathbb{R}_{\geq 0}\mathcal{I}(n,2)$ the smallest positive homogeneous set containing $\mathcal{I}(n,2)$.

Proposition (Averkov-S'22) We have

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Corollary The space $\overline{\mathbb{R}_{\geq 0}\mathcal{I}(4,2)}$ is completely described by the Plücker-type inequalities.

Other directions

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- Do Plücker-type inequalities exist in higher dimensions?

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- What inequalities for $\mathcal{PV}(8, 2)$ are we missing?
- Do Plücker-type inequalities exist in higher dimensions?
- Investigate relations between $\mathcal{PV}(n, d)$ and square-free Lorentzian polynomials.

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- Is $\mathcal{PV}(n, 2)$ completely described by the Plücker-type inequalities for $n = 5, 6, 7$?
- What inequalities for $\mathcal{PV}(8, 2)$ are we missing?
- Do Plücker-type inequalities exist in higher dimensions?
- Investigate relations between $\mathcal{PV}(n, d)$ and square-free Lorentzian polynomials.

Thank you!