Multiplier Ideals of Certain Binomial Ideals

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The definition of multiplier ideals

Definition

Let $X$ be a smooth variety $X$ over $\mathbb{C}$ and let $\mathcal{I}$ be a nontrivial coherent ideal sheaf on $X$. Choose a log resolution $\pi : Y \rightarrow X$ of $\mathcal{I}$. Let $\mathcal{F}$ be the normal crossings divisor such that $\mathcal{I} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-\mathcal{F})$. Then for all positive real numbers $\lambda$, the multiplier ideal sheaf of $\mathcal{I}^\lambda$ on $X$ is

$$\mathcal{J}(\mathcal{I}^\lambda) = \pi_* \mathcal{O}_Y \left( K_{Y/X} - \lfloor \lambda \mathcal{F} \rfloor \right)$$

$$= \bigcap_E \left\{ f \in \mathcal{O}_X \mid \text{ord}_E(f) \geq \lfloor \text{ord}_E(\mathcal{I}) \lambda - \text{ord}_E(\mathcal{J}_{\mathcal{O}_E/\mathcal{O}_X}) \rfloor \right\}$$

where the intersection ranges over the exceptional divisors $E$ and $\mathcal{J}_{\mathcal{O}_E/\mathcal{O}_X}$ is the Jacobian ideal of $\mathcal{O}_E$ over $\mathcal{O}_X$. 

Howald’s theorem

Theorem (Howald 2001)

Let $X = \mathbb{A}^n$. If $I \subset \mathbb{C}[x_1, x_2, \ldots, x_n] = \mathbb{C}[x]$ is a monomial ideal with Newton polyhedron $\text{Newt}(I)$, then, for all $\lambda > 0$,

$$\mathcal{J}(I^\lambda) = \{ x^v \in \mathbb{C}[x] | v + 1 \in \text{the interior of } \lambda \text{Newt}(I) \}.$$ 

Note that the fan of the normalized blowup of $I$ is the normal fan to its Newton polyhedron $\text{Newt}(I)$. So, Howald’s formula tells us that, for monomial ideals,

$$\mathcal{J}(I^\lambda) = \bigcap_{\nu} \{ f \in \mathbb{C}[x] | \nu(f) \geq \lfloor \nu(I)\lambda - \nu(J_{R\nu/\mathbb{C}[x]}) \rfloor \}$$

where the intersection ranges over the Rees valuations of $I$. 
Notation & setup

We assume the standard notation of toric geometry as found in Fulton’s book.

Let $S_\tau$ be a normal affine semigroup and let $\varphi : S_\tau \to \Gamma$ be a surjective homomorphism of affine semigroups. Let $Z^\Gamma \subset X_\tau$ be the corresponding inclusion of (not necessarily normal) affine toric varieties. Let $\sigma = \tau \cap (\ker(\varphi \otimes \mathbb{R}))^\perp$, let $X_{\sigma,N_\sigma}$ be the toric variety associated to the cone $\sigma$ and the lattice $N_\sigma = N \cap \mathbb{R}\sigma$.

[We assume $\Gamma$ has a trivial unit group. We also assume $Z^\Gamma$ is not contained in a smooth toric hypersurface of $X_\tau$. We can use adjunction/inversion of adjuntion to reduce to this case when computing multiplier ideals.]
González Pérez & Teissier’s theorem on partial embedded resolutions

Theorem (González Pérez & Teissier 2002)

If \( Z^Γ \) is not contained in a smooth toric hypersurface of \( X_τ \) and \( Σ \) is any subdivision of \( τ \) containing \( σ \), then:

1. The strict transform \( Z^Γ_Σ \) of \( Z^Γ \) via the toric morphism \( π_Σ : X_Σ \to X_τ \) is contained in the open affine subvariety \( U_σ \subset X_Σ \). It is isomorhic to \( X_σ, N_σ \). And, the restriction \( π_Σ|_{Z^Γ_Σ} : Z^Γ_Σ \to Z^Γ \) is the normalization map.

2. If \( Σ' \) is any regular subdivision of \( Σ \), then \( π_{Σ'} : X_{Σ'} \to X_τ \) is an embedded resolution of \( Z^Γ \subset X_τ \).

In general, \( \dim σ = \dim Z^Γ \) and the open set \( U_σ \cong T \times X_σ, N_σ \) where \( T \) is a torus of complementary dimension. Moreover, \( Z^Γ_Σ \) meets the torus at the unit \( 1 \in T \).
Two examples

Example (Hypersurfaces in $\mathbb{A}^n$)

When $Z^\Gamma$ is a hypersurface in $\mathbb{A}^n$, $\ker(\varphi \otimes \mathbb{R})$ is a hyperplane in $N_\mathbb{R}$. So, we can take $\Sigma$ to be the fan obtained by slicing the positive orthant along that hyperplane. This $\Sigma$ is the fan of the normalized blowup of the term ideal of the ideal $I_{Z^\Gamma}$.

Example (Curves in $\mathbb{A}^n$)

Let $\varphi : \mathbb{N}^n \rightarrow \Gamma$ be given by the matrix $[a_1 \ a_2 \ \ldots \ a_n]$. Then $Z^\Gamma \cong \text{Spec} \mathbb{C}[t^{a_1}, t^{a_2}, \ldots, t^{a_n}]$, $X_\tau = \mathbb{A}^n$, $\sigma = \mathbb{R}_{\geq 0} \ [a_1 \ a_2 \ \ldots \ a_n]$, $N_\sigma = \mathbb{Z} \ [a_1 \ a_2 \ \ldots \ a_n]$, $X_{\sigma, N_\sigma} \cong \mathbb{A}^1$, we can take $\Sigma$ to be the stellar subdivision along the ray $\sigma$. Here, $X_\Sigma$ is the blowup of $(x_1^{A/a_1}, x_2^{A/a_2}, \ldots, x_n^{A/a_n})$ where $A$ is any common multiple of the $a_i$s.
A monomial space curve formula

Theorem (—)

Let $C = \text{Spec } \mathbb{C}[t^a, t^b, t^c] \subset \mathbb{A}^3 = \text{Spec } \mathbb{C}[x, y, z]$ and assume none of $a$, $b$ and $c$ are contained in the numerical semigroup generated by the other two. Let $I = I_C$ and let $\{f_1, f_2, \ldots\}$ be a minimal binomial generating set for $I$ written in nondecreasing degree when $x$, $y$ and $z$ have degrees $a$, $b$ and $c$ respectively. Then, we need only the $I$-adic valuation and the valuation $\nu$ to compute the multiplier ideals, where $\nu$ is that valuation with generating sequence $x \mapsto a$, $y \mapsto b$, $z \mapsto c$ and $f_1 \mapsto \deg(f_2)$.

$$\mathcal{J}(I^\lambda) = I^{\lfloor \lambda - 1 \rfloor} \cap \{ f \in R \mid \nu(f) \geq \lfloor \nu(I) \lambda - \nu(J_{R, \nu}/R) \rfloor \}$$

where $\nu(I) = \deg(f_2)$ and $\nu(J_{R, \nu}/R) = a + b + c - 1 + \deg(f_2) - \deg(f_1)$. 
The (3,4,5)-curve

Example

Let $C = \text{Spec} \mathbb{C}[t^3, t^4, t^5] \subset \mathbb{A}^3 = \text{Spec} \mathbb{C}[x, y, z]$. So, $I = (y^2 - xz, x^3 - yz, z^2 - x^2y)$. Let $a = (x^{20}, y^{15}, z^{12})$. We start with the partial embedded resolution of $C$ to get $X_\Sigma$. That is, we blow up $a$, or equivalently, we subdivide along $[3 \ 4 \ 5]$. We obtain a toric variety such that the strict transform of the curve is smooth and contained in the smooth locus. However, there is an embedded component supported on the intersection of the exceptional divisor and the strict transform of $V(y^2 - xz)$. 
The partial embedded resolution of the \((3,4,5)\)-curve
The partial embedded resolution of the \((3,4,5)\)-curve

\[ \chi_\Sigma \]

- \((0,1,0)\)-divisor
- \((1,0,0)\)-divisor
- \((0,0,1)\)-divisor
- \((3,4,5)\)-divisor

- embedded component
- strict transform
Example (The (3,4,5)-curve, continued)

On the open affine $U_\sigma$, $A_\sigma = \mathcal{O}_U = \mathbb{C} \left[ \frac{y}{x}, \left( \frac{y^2}{xz} \right)^{\pm 1}, \left( \frac{x^3}{yz} \right)^{\pm 1} \right]$ and

$I = \left( \left( \frac{y}{x} \right)^8 \left( \frac{y^2}{xz} - 1 \right), \left( \frac{y}{x} \right)^9 \left( \frac{x^3}{yz} - 1 \right) \right)$. In particular, $I$ is monomial in $\frac{y}{x}$, $\frac{y^2}{xz} - 1$ and $\frac{x^3}{yz} - 1$. Unfortunately, we don’t have this good behavior at the other two points on the embedded component. So, we take the normalized blowup of the total transform of $(y^2, xz)$. This creates a new toric variety $X_{\Sigma'}$ that is a partial embedded resolution of both $C$ and $V(y^2 - xz)$. That is, any regular subdivision of $\Sigma'$ yields an embedded resolution of both $C$ and $V(y^2 - xz)$. 
The second blowup

\[ X_\Sigma \]

- **(0,1,0)-divisor**
- **(1,0,0)-divisor**
- **(3,4,5)-divisor**
- **(0,0,1)-divisor**

**Strict transform**

**Embedded component**
The second blowup

\( X_{\Sigma} \)

\( (2,1,0) \)-divisor

\( (0,1,0) \)-divisor

\( (0,1,2) \)-divisor

\( (0,0,1) \)-divisor

\( (1,0,0) \)-divisor

strict transform

embedded component
Monomialization (toroidalization) has been acheived

Example (The (3,4,5)-curve, continued)

On the 2-cone $\mathbb{R}^2_{\geq 0} \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix}$, the coordinate ring is $\mathbb{C} \left[ z, y, \frac{y^2}{z}, \frac{y^3}{z^2}, \frac{y^4}{z^3}, \frac{y^5}{z^4}, \left( \frac{xz}{y^2} \right)^\pm 1 \right]$. And, the point of interest has maximal ideal $\left( z, y, \frac{y^2}{z}, \frac{y^3}{z^2}, \frac{y^4}{z^3}, \frac{y^5}{z^4}, \frac{xz}{y^2} - 1 \right)$. On any affine chart around this point such that $\frac{x^3}{yz} - 1$ does not vanish, $I = \left( y^2 \left( \frac{xz}{y^2} - 1 \right), yz \right)$, is monomial in $\left\{ z, y, \frac{y^2}{z}, \frac{y^3}{z^2}, \frac{y^4}{z^3}, \frac{xz}{y^2} - 1 \right\}$.

Behavior at the other point not in $U_\sigma$ is similar.
The formula for the multiplier ideals of the (3,4,5)-curve

Example (The (3,4,5)-curve, continued)

Now that the total transform is monomial everywhere, it suffices to look to the Rees valuations of the total transform. For that, we need only consider \( \left( \left( \frac{y}{x} \right)^8 \left( \frac{y^2}{xz} - 1 \right) , \left( \frac{y}{x} \right)^9 \left( \frac{x^3}{yz} - 1 \right) \right) \) and Howald’s formula since the generic points of the components of the total transform of \( I \) are in \( U_\sigma \). The valuations we get are the \( I \)-adic one \( \frac{y^2}{xz} - 1, \frac{x^3}{yz} - 1 \mapsto 1 \) and \( \nu : \frac{y}{x}, \frac{y^2}{xz} - 1 \mapsto 1 \). So,

\[
\mathcal{J}(I^\lambda) = I(\lfloor \lambda - 1 \rfloor) \cap \{ f \in \mathbb{C}[x, y, z] \mid \nu(f) \geq \lfloor 9\lambda - 12 \rfloor \}
\]

where \( \nu \) is given by the generating sequence \( x \mapsto 3, y \mapsto 4, z \mapsto 5 \) & \( y^2 - xz \mapsto 9 \).
The partial embedded resolution of the \((3,4,5)\)-curve and \(V(y^2 - xz)\)
The partial embedded resolution of the $(3,4,5)$-curve and $V(y^2 - xz)$
The partial embedded resolution of the (3,4,5)-curve and $V(y^2 - xz)$
A cross-section of the fan
Thank you